

## Who are controlling the DERs? Increasing DER hosting capacity through targeted modeling, sensing, and control

Final Project Report

T-64

**Power Systems Engineering Research Center** 

Empowering Minds to Engineer the Future Electric Energy System

# Who are controlling the DERs? Increasing DER hosting capacity through targeted modeling, sensing, and control

**Final Project Report** 

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#### **Executive Summary**

Distributed energy resources (DERs) offer many opportunities for providing grid services but may also cause issues in distribution grid operations. The overall goal of this project was to develop modeling, analysis and optimization techniques for systems that improves our understanding and ability to control systems with many distributed energy resources (DERs). These new techniques help us understand how DERs interact with the electric distribution grid, and how to assess the variability, control opportunities and sensing needs that arise with higher penetrations for DERs.

More specifically, the project had three main goals, namely to (1) better characterize the uncertainty associated with DER power injections and control, (2) assess the optimal placement of voltage sensors in the distribution grid and (3) identify secure ranges for DER control by third-party aggregators. Progress towards these goals was achieved in three integrated tasks, with frequent interactions and discussions between the team members. Given the distinct goals of each task, the research findings of this report is presented in three separate parts corresponding to the three tasks of the project. The goals and research findings of each task is included below.

## Part I: From load modeling and uncertainty characterization to distribution system operational scheduling and optimal inverter control

**Part I** includes two chapters. First chapter presents approaches for load modeling and uncertainty characterization. Second chapter proposes two advanced approaches for distribution system operational scheduling and DER control and optimization.

Load modeling and uncertainty characterization: This project designed and developed load forecasting and advanced aggregation strategies based on machine learning approaches. Artificial Neural Network is used for aggregated load forecasting while the forecast errors were evaluated and characterized as uncertainty of forecasted load. Fitness of load uncertainty to known probability distributions were checked by Shapiro–Wilk test. Next, to enable assessment of demand side flexibility, appliance-level load monitoring and forecasting techniques were investigated. The knowledge of distribution load characteristics enables more effective management of DER-enriched distribution systems. For example, knowing the percentage of air conditioning load assists operators in understanding load flexibility (for direct load control or other demand response schemes). As another example, discovery of connected electric motor load enables better assessment of voltage stability concerns, i.e., fault-induced delayed voltage recovery. Another example is home energy management, which is performed at the appliance-level. To this end, the team has developed an optimization approach along with artificial intelligence (AI)-based approaches, namely conditional hidden semi-Markov model. The accuracy of these models was tested using actual historical smart meter data.

<u>Distribution system operational scheduling and optimal DER control:</u> This project proposes two distribution operational scheduling frameworks based on two different unbalanced three-phase distribution AC optimal power flow models: A) a novel current-voltage rectangular approach, and B) a two-stage second-order cone programming-based approach. Both of these algorithms accurately capture unique characteristics of distribution systems, namely mutual impedances, single-phase to three-phase feeders and laterals, and unbalanced nature. We tested both of these algorithms using data of an actual distribution feeder and corresponding load and PV generation from an electric utility. The comparison of power flow results of these models with OpenDSS

(EPRI's distribution power flow analysis tool) confirms the accuracy and effectiveness of these algorithms in capturing unique characteristics of distribution systems. Moreover, both these proposed unbalanced AC optimal power flow models are enhanced to account for Volt-Var controller of PV smart inverters based on the characteristic that is outlined in IEEE Standard 1547-2018. Moreover, these algorithms are further enhanced to optimize droop characteristics of Volt-Var controllers within the allowable range, which is set by IEEE Standard 1547-2018. The results of testing these algorithms on an actual distribution feeder show that optimizing Volt-Var controller settings and droop characteristics of smart inverters substantially enhance distribution voltage management and reduces the need for active power curtailment of DERs to maintain voltage within the acceptable range as set by ANSI standard. This is a critical contribution of this project as distribution system voltage management is deemed to be more challenging with high penetration levels of DERs.

#### **Project Publications:**

- [1] Mingyue He and Mojdeh Khorsand, "Residential Appliance-level Consumption Modeling and Forecasting via Conditional Hidden Semi-Markov Model", *the* 54<sup>th</sup> North American Power Symposium, 2022.
- [2] Mingyue He, Zahra Soltani, Mohammad Ghaljehei, Masoud Esmaili, Shanshan Ma, Mengxi Chen, Mojdeh Khorsand, Raja Ayyanar, and Vijay Vittal, "A SOCP-based ACOPF for Operational Scheduling of Three-phase Unbalanced Distribution Systems and Coordination of PV Smart Inverters," *IEEE Transactions on Power Systems*, under review.
- [3] Zahra Soltani, Shanshan Ma, and Mohammad Ghaljehei, "Optimal Scheduling of Distributed Energy Resources Considering Volt-VAr Controller of PV Smart Inverters," the 11<sup>th</sup> Bulk Power Systems Dynamics and Control Symposium, July 2022.

#### **Student Theses:**

- [1] Mingyue He, "Distribution system operation and control enhancement: from load forecasting and DER aggregation to distribution operational scheduling," *PhD Dissertation, Arizona State University,* expected May 2024.
- [2] Zahra Soltani, "Sensor Enabled Advanced Distribution Management System: From Topology Processor and State Estimation to Distributed Energy Resources Scheduling," *PhD Dissertation, Arizona State University,* expected May 2023.

## Part II: Bilevel Optimization Approaches for Locating Sensors in Distribution Systems

In **Part II**, we developed an algorithm for finding optimal sensor locations and alarm thresholds for rapid identification of voltage violations. This algorithm solves a bilevel optimization problem which minimizes the number of sensors and the number of false alarms (upper-level problem) while making sure that there are no voltage violations that go undetected (lower-level problem). Since this bilevel problem is difficult to solve directly, we developed several approaches for addressing its computational challenges. We first addressed the power flow nonlinearities in the lower-level problem by developing conservative linear approximations of the power flow equations. These approximations provide overestimates and underestimates of the nonlinear power flow equations over the relevant operational range. By replacing the nonlinear power flow equations with these conservative linear approximations, we can ensure that the resulting sensor locations and alarm thresholds are sufficient to identify any constraint violations that might occur for any power injections within a specified range of power injection fluctuations. The accuracy of these conservative linear approximations is closely connected to the quality of the sensor locations and alarm thresholds that result from solving the reformulated bilevel optimization problem. We therefore spent significant effort in developing methods for improving their accuracy. These methods are discussed in detail in this report.

Even after replacing the power flow equations with these conservative linear approximations, the sensor placement problem remains computationally challenging due to its bilevel nature. As we show with several numerical test cases, standard techniques for reformulating the sensor placement problem to a single-level formulation that can be addressed with existing solvers yield optimization problems that are only tractable for small test cases. Accordingly, the team exploited structure specific to this problem to obtain single-level mixed-integer programming formulations that can be solved with commercial software packages like Gurobi. Key insights in these reformulations include the importance of discretizing the sensor thresholds and a tailored dualization approach that avoids introducing unnecessary additional discrete variables, as would be required using traditional methods for bilevel problems.

The resulting sensor placement reformulations require substantially less computational time than standard reformulation techniques, thus enabling us to scale up to problems with tens to hundreds of buses. Furthermore, we developed extensions to these reformulations that consider the possibility of multiple network topologies, with the results still showing strong computational scalability characteristics.

To improve the sensor placements resulting from this bilevel problem formulation, we developed a post-processing technique that reduces the number of false alarms. Using an approximate gradient approach, this post-processing technique adjusts the sensor thresholds to make them less conservative while not introducing false negatives (missed alarms). The combination of the bilevel problem reformulation and this post-processing technique allows us to compute sensor locations and alarm thresholds that have a small number of false alarms and no missed alarms, as validated numerically using out-of-sample testing.

Next steps for this work include modeling extensions to consider sensors with more general capabilities and more detailed representations of the distribution network and associated devices. Complementing the test cases considered in this report, we also aim to demonstrate the developed techniques on actual large-scale distribution system data sets in order to assess their scalability and quality of the resulting solutions. Finally, we intend to apply the mathematical reformulation techniques developed in this project to other challenging optimization problems beyond the sensor placement problem considered in this report.

#### **Project Publications:**

- P. Buason, S. Misra, and D.K. Molzahn, "A Sample-Based Approach for Computing Conservative Linear Power Flow Approximations," *Electric Power Systems Research*, vol. 212, no. 108579, November 2022. Presented at 22nd Power Systems Computation Conference (PSCC), June 27 - July 1, 2022.
- [2] P. Buason, S. Misra, and D.K. Molzahn, ``A Data-Driven Method to Identify Critical Locations for Sensors in Distribution Grids," in preparation.

#### **Student Theses:**

[1] Paprapee Buason, "Addressing Power Flow Nonlinearities in Bilevel Optimization Problems with Applications to Optimal Sensor Placement in Electric Distribution Systems," *PhD Dissertation, Georgia Institute of Technology, School of Electrical and Computer Engineering*, December 2024 (expected).

#### Part III: Identifying Secure Operation Ranges For DER Control by Third-Party Aggregators

In **Part III**, we focused on challenges associated with coordinating control activation from DERs across multiple different entities, including third-party DER aggregators who are able to dispatch the DERs through contracts with DER owners, the distribution system operator (DSO) who is in charge of making sure that no grid constraints are violated in the distribution grid, and the independent system operator who needs to contract flexibility services from the third-party aggregator while respecting distribution grid constraints. Our research takes the perspective of a DSO who aims to identify a secure flexibility region for DER active power control within the distribution grid. We envision that the this secure flexibility region can be used as feeder-level capacity constraints in the real-time market clearing to ensure that DER flexibility provided by third-party aggregators will not violate any distribution level constraints. Specifically, the ISO could use these flexibility ranges to provide a simplified view of distribution grid constraints when clearing bids from DER aggregators with resources located inside the distribution grid. This research thus proposes one solution for integration of aggregated DER resources in electricity market clearing, as mandated by FERC Order 2222.

The key innovation of this work with respect to existing literature is that we allow entities other than the distribution system operator to make decisions on how to dispatch individual DERs within the feeder. Existing work has either assumed that the DSO can directly dispatch active power from DERs or have provided flexibility ranges for each DER. However, such assumptions place unrealistic assumptions on coordination and information exchange between third-party aggregators and DSOs, such as the sharing of grid models or confidential bid information. These problems become particularly acute when there is more than one aggregator with resources located inside the feeder. To circumvent these issues, we assume that third-party aggregators decide how individual DERs are activated to achieve a certain control outcome, without further coordination with the DSO or other aggregators within the feeder. The DSO, on the other hand, are only able to provide limits on the total amount of DER control activation the feeder is able to provide. The goal of our research is to identify how large this total amount is allowed to be.

To identify secure flexibility ranges for this case, we have to assume that these other entities (such as third-part aggregators) could potentially choose dispatch DERs in a harmful way. We therefore formulate and solve the problem as an adversarial bilevel program, where the upper level problem (representing the DSO) maximizes the range of flexibility available from the feeder, while the lower level problem (representing the joint control activation across all DERs) enforces that the maximum and minimum achievable voltage magnitudes remain within safe bounds. Part III of this report presents our formulation of a linearized bilevel problem, discusses how to obtain a tractable single level reformulation and provides numerical results.

As part of our research, we identified that the secure flexibility region can be significantly increased by incorporating some simple rules for DER control activation:

- First, we identify that we need to ensure that **all DERs are controlled in the same direction**, i.e., the DERs can either increase or decrease their set-points relative to the current operating point. This prevents situations in which some DERs increase their active power injections while others decrease their injections, leading to potential voltage problems even with very small total active power control.
- Second, we evaluate the impact of different modes of reactive power control including setting a constant reactive power injection, determining the power factor or controlling the active power droop. We identify that several different modes can provide good results. However, it is vitally **important that the DSO decides how reactive power is controlled**, rather than allowing individual DERs to choose (potentially harmful) reactive power setpoints.
- To link with the work done in Task 2 and 3, we also considered how limited access to measurements and uncertainty regarding the current operating point of the system can be reflected in this optimization problem. In our case study, we observe that it is possible to obtain wide flexibility ranges, even with limited real-time observability, although further investigations are needed to confirm these results more widely.

Beyond the main conclusions above, we have also run experiments to confirm that the accuracy of our linearized optimization model is acceptable.

Immediate next steps for this work include identification of the most practical DER reactive power control modes and more detailed discussion of the case without real-time measurements. More significant, but practically important aspects include the consideration of multi-period problems (i.e., how flexibility ranges may change as DER control capability and system set-points evolve over time) as well as forecasting of flexibility ranges for future operating periods to support, e.g., day-ahead market clearing. We would also like to test our methods on realistic distribution feeders to address possible issues associated with data access and confidential information.

#### **Project Publications:**

- K. Girigoudar and L. A. Roald, "Linearized Three-Phase Optimal Power Flow Models for Distribution Grids with Voltage Unbalance," *IEEE Conference on Decision and Control* (CDC), December 2021, pp. 4214-4221, doi: 10.1109/CDC45484.2021.9683780.
- [2] K. Girigoudar and L. A. Roald, "Identifying Secure Operating Ranges for DER Control using Bilevel Optimization," *submitted*.

#### **Student Theses:**

[1] Kshitij Girigoudar. "Optimization of Unbalanced Distribution Grid Operations using Distributed Energy Resources". *Phd Dissertation, University of Wisconsin Madison, Department of Electrical and Computer Engineering*, August 2022

## Part I

## Load and DER modeling and uncertainty characterization

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## Nomenclature

### Acronyms

ACOPF	AC Optimal Power Flow
ANN	Artificial Neural Network
CHSMM	Conditional Hidden Semi-Markov Model
DERs	Distributed Energy Resources
DNN	Deep Neural Network
DR	Demand Response
DSMS	Demand-Side Management Systems
HEMS	Home Energy Management Systems
HMM	Hidden Markov Model
MNLR	Multinomial Logistic Regression
MISCOP	Mixed-Integer Second-Order Cone Programming
VRT	Voltage Regulation Transformers
VVC	Volt-VAr Controllers
OPF	Optimal Power Flow
PV	Rooftop Solar Photovoltaic
Q-V	Reactive power-Voltage
SCC21	Standards Coordinating Committee 21
SOCP	Second-Order Cone Programming

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#### 1. Load and DER Modeling and Uncertainty Characterization

#### **1.1 Introduction**

Distributed energy resources (DERs) offer many opportunities for providing grid services but may also cause issues in distribution grid operations. For example, the overvoltage issue can be caused by adding DERs. However, DERs provide operational flexibility by adjusting power production/consumption, e.g., demand response and reactive power support by smart inverters. A distribution utility is able to control the active and reactive power outputs of some DERs to improved power quality. The goal of this project is to develop the modeling, monitoring, and control techniques for systems with many different DERs, with potentially conflicting control objectives.

Residential building energy consumption is responsible for the total energy consumption in the United States from about 14.8 to 20.8 quadrillion (British thermal units) from 1975 to 2020, respectively, according to the United States Energy Information Administration [1]. The residential demand-side management has become an integral element of the entire grid over the last two decades due to the growth of residential demand, the development of the smart grid, and the increasing penetration of distributed renewable resources. Demand-side management systems (DSMS) and home energy management systems (HEMS) enhance residential energy efficiency and are critical for maintaining the balance between demand and generation from distributed energy resources (DERs) [2]-[3]. Recently, the appliance or device-level load modeling and forecasting have attracted great attention for DSMS and HEMS to understand the characteristics of the residential demand to achieve energy utilization efficiency and energy savings. Some demand response (DR) programs of electric utilities are designed for some specific appliances [4]-[7]. Those designed DR programs make non-intrusive appliance-level load monitoring an important task for electric utilities to enhance the assessment of the impact of DR on their system. However, non-intrusive appliance-level load monitoring are challenging because of the inherent variability of exogenous environmental factors and the intrinsic uncertainty of human behavior [8]. A non-intrusive appliance-level load monitoring technique is required for DSMS and HEMS to assess the load flexibility in the residential household, enhance observability of the distribution system, and support efficient grid operation to account for the uncertainty and variability of the distribution system due to the increase of DERs.

#### **1.2** Literature review

#### 1.2.1 Load uncertainty modeling and characterization

Load uncertainty modeling and characterization have been studied. For most of the previous work, the uncertainty of load and DERs is modeled by a probability distribution function based on a onesize-fit-all approach. References [10]-[15] use Gaussian distribution to model the load uncertainty. References [16]-[17] consider Gaussian mixture model for uncertainty modeling. References [18]-[19] use a flat range (i.e., min and max) of load forecasting errors to model uncertainty in a robust optimization problem. Some of the prior work introduce machine learning algorithms to learn the uncertainty of the load [20]-[22]. Reference [20] utilizes Bayesian deep learning to capture residential net load uncertainty. Reference [21] uses the trained quantile regression model to represent load uncertainty. Reference [22] fits load forecasting errors into a hermite polynomial function for load uncertainty. Fuzzy set theory is used in reference [23] to represent the load uncertainty in optimization model. Euclidean norm is used to represent the load variation under different certain degree of uncertainty in [24]. Trajectory sensitivity approach is used in [25] to account for load modeling uncertainties for voltage stability problem.

The project team proposes an approach to aggregate customers, utilizes supervised machine learning algorithm to learn customers' consumption pattern, and uses Gaussian distribution to model the load uncertainty. The hypothesis of using Gaussian distribution is checked by Shapiro–Wilk test.

#### **1.2.2** Non-intrusive appliance-level load forecasting and monitoring

Many previous papers have been published for appliance-level load modeling and forecasting with the utilization of machine learning techniques. References [26]-[27] use deep neural network (DNN) for short-term appliance-level power forecasting and appliance status identification. The main drawback of reference [26]-[27] is that DNN needs large storage memory for historical data and takes a very long time to train a model, which makes DNN impractical to be used for a large number of residential appliances. Hidden Markov model (HMM) is one common machine learning algorithm used for home individual appliances consumption modeling and forecasting. Most household appliances have finite discrete operating modes, and the current appliance consumption is independent of the early past status of the appliance, which fits HMM structure well. Reference [28] uses HMM to identify the household appliances with two operating states (on/off). The authors in [29] infer the magnitude, duration of each state, and variability of household consumption by utilizing HMM framework. Reference [30] proposes a modified HMM for detecting and estimating household individual appliance loads from the aggregated power. Reference [31] proposes a hierarchical HMM framework to model home appliances with distinct power consumption profiles, such as washing machines and dishwashers. References [28]-[31] take advantage of HMM structure to model characteristics of household appliances. However, the relationship between appliances' state, duration of each state, and consumption are even more complex, and the behavior of household appliances also depends on exogenous environmental variables such as temperature and time of the day [8]. Those characteristics make the original HMM framework insufficient for appliance-level modeling and forecasting. A Conditional Hidden Semi-Markov Model (CHSMM) is proposed in [8] to account for the drawback of HMM framework.

A number of work has been focused on non-intrusive appliance-level load forecasting and monitoring problem. Reference [32] uses active and reactive power consumption change of the appliance to identify the appliance on/off. Reference [33] introduces penalty calculation to filter out some appliance combinations and has a correction step after getting preliminary result. Reference [34] uses voltage and current information to get fundamental and harmonic frequencies to estimate magnitude and phase of load admittance to identify the appliance. Reference [35] utilizes dynamic time warping to identify appliance status with correction step. Reference [36]-[37] consider different penalty term to achieve higher accuracy of their load disaggregation model. Reference [38] uses a machine learning algorithm, i.e., factorial hidden Markov model, to

disaggregate load. To achieve the non-intrusive appliance-level load forecasting and monitoring, the project team utilizes CHSMM with an optimization-based load disaggregation model.

#### 1.3 Load and DERs uncertainty modeling and characterization

In this task, supervised machine learning algorithms are trained and used to predict aggregated DERs and load based on their prior net power exchange and similar day's consumption, and temperature information. The approach is summarized in a flowchart as shown in Figure 1.1. In the beginning of the approach, prosumers are aggregated into different groups. Then, machine learning models are trained via aggregated prosumers' historical data. The training features include historically prior off-peak consumption and similar day's consumption, and corresponding temperature. The trained model with the highest accuracy is selected to predict the consumption baseline of on-peak periods. Afterward, the DR reductions and prosumers' response for different prosumers' groups are computed based on the obtained accurate consumption baseline and prosumers consumption data. Since prosumers' involvement and response are dynamic, the uncertainty needs to be evaluated for different time of the day. Eventually, prosumers' uncertainty can be evaluated based on the forecasting differences by certain probability distribution functions. In this work, Gaussian distribution is used to fit prosumers' response for different periods to represent the estimated DR reductions and prosumers' response of a DR program for the aggregated prosumer groups.



Figure 1.1 The flowchart of the proposed uncertainty modeling and characterization

The proposed algorithm is examined on a modified IEEE 33-bus test distribution system to which DR programs are added. The aggregated prosumers' consumption data from a local electric utility company is scaled based on the existing load in the IEEE 33-bus system. Without loss of generality, it is assumed that all prosumers connected to a bus belong to a similar prosumers' group.

The hypothesis of using Gaussian distribution to model the uncertainty of prosumers' response is evaluated. The obtained forecasting differences are used to fit a Gaussian distribution shown in Figure 1.2. The means of the obtained forecasting differences are all close to zero. As can be seen, the Gaussian distribution can be a good choice to model the uncertainty of the prosumers' response. In addition, the quantile-quantile plot is used to check the hypothesis of Gaussian distribution in Figure 1.2. The quantile-quantile plot is a graphical technique for determining if two data sets come from populations with a common distribution. If the two distributions being compared are similar, the points in the quantile-quantile plot will approximately lie on the identity line y = x.

As can be seen in Figure 1.3, the prosumers' response data point is lying on the dashed line, which shows that the prosumers' response follows Gaussian distribution.



Figure 1.2 Uncertainty with fitted Gaussian distribution



Figure 1.3 Quantile-quantile plot for assumption of Gaussian distribution

To achieve a more general conclusion, Shapiro–Wilk test is used to check the hypothesis of Gaussian distribution for all prosumers' groups. Shapiro–Wilk test is the most powerful metric for evaluating the hypothesis of Gaussian distribution. After using Shapiro–Wilk test to the hypothesis of Gaussian distribution for all prosumers' groups, we find that 99.6% of 15-min samples and 97.7% of hourly samples pass the Shapiro–Wilk test. This result further confirms that the hypothesis of using Gaussian distribution to model prosumers' response uncertainty is correct.

The table shows the averaged accuracy of the trained model by aggregated prosumers' data. As can see in Table 1.1, the accuracy is high for the trained models.

#### Table 1.1 Averaged accuracy of the trained model

Averaged trained model accuracy (NRMSE)	Averaged trained model accuracy (R2)
13.8%	0.90

Figure 1.4 and Figure 1.5 show examples of the uncertainty at bus 7 on August 7th. The blue area is the 95% confidence level of Gaussian distribution. The hourly and 15-min prosumers' response uncertainty are considered as shown in Figure 1.4 and Figure 1.5. In this task, the obtained uncertainty information is passed to Task 2 and Task 3.



Figure 1.4 One example of showing the hourly load uncertainty (95% confidence)



Figure 1.5 One example of showing the 15-min load uncertainty (95% confidence)

The demand response uncertainty is evaluated by the proposed approach, which is shown in Figure 1.6. The uncertainty of DR is different for different groups, hours of the day, and price plans. As can be seen, the random aggregation fails to accurately capture prosumer behavior and response and leads to the highest uncertainty (see dark red curve). However, by using better aggregation strategies, the uncertainty of demand response is reduced, e.g., the orange curve. Moreover, with integration of more and more smart grid technologies, the differences shown in the Figure 1.6 would be more severe.



Figure 1.6 Demand response uncertainty using the proposed approach

#### 1.4 Non-intrusive appliance-level load forecasting and monitoring

#### 1.4.1 Non-intrusive appliance-level load forecasting

HMM provides a framework to describe the pattern of individual appliance consumption. Most residential household appliances have one or several operating modes, which can be modeled as finite discrete states. The current energy consumption of the appliance can be assumed to be dependent on the current state of the appliance. In addition, the current state of the appliance is independent of the early status of the appliance in the past, which follows the Markov process that is the key assumption of HMM. Therefore, HMM framework seems a potential choice for household appliances modeling and forecasting.

The graphical illustration of the HMM framework for appliance-level modeling and forecasting is shown in Figure 1.7. The appliance energy consumptions from period 1 to period k are denoted as  $Y = \{y_1, y_2, ..., y_k\}$ , and the latent states of the appliance are represented as  $X = \{x_1, x_2, ..., x_k\}$ . The state-space of the appliance is introduced to refer to the operating modes of the appliance. The state space is denoted as  $S = \{s_1, s_2, ..., s_n\}$ , where n is the number of states. For example, A/C has two operating modes (on/off), and the state space of the A/C is {on, off}. Under the HMM structure, the observable outcomes are only affected by the current state. For instance, the outcome of  $y_k \in Y$  at period k only depends on the state of appliance  $x_k \in X$  at period k. The outcomes and states of the appliance at any previous period cannot affect the outcome of  $y_k \in Y$  at period k. In addition, the state  $x_{k+1}$  is only dependent on the previous state  $x_k$  due to the assumption of the Markov process.



Figure 1.7 Graphical illustration of HMM

The HMM framework has the ability to track the energy usage patterns of household appliances. However, some characteristics of energy consumption of appliances are not fully addressed by the original HMM structure. For instance, the appliance may stay in one operating mode for certain periods after the transition of operating modes, which means that the current state of an appliance remains constant until the duration of the current state expires. This also implies that the Markov transition to a new state only happens when the duration of the state ends. In addition, for some appliances, their current state can depend on the previous state and the duration of the previous state. The duration of the current state can also depend on the previous state, the duration of the previous state, and the current state. For example, the duration of operating modes of a refrigerator can vary significantly by its previous operating mode and duration [8]. Moreover, some household appliances can be easily affected by some exogenous factors such as temperature and time of the day. For instance, the outdoor temperature can be one exogenous variable for on/off states of the A/C. Higher temperatures can lead to a higher chance for A/C to stay in cooling mode. In addition, the cooling/heating energy consumption of some A/C can also be affected by the temperature. Furthermore, the state and the duration of the state for a refrigerator may be affected by the temperature. In general, a refrigerator has four operating modes, which are compressor off, compressor on, ice making, and defrosting. A refrigerator may have more chances to stay in the compressor off or defrosting mode in cold weather. To address those important characteristics of the appliance, a modified version of HMM, namely CHSMM, is used for the appliance-level modeling and forecasting.

CHSMM is an extension of the concept of HMM to consider the variable duration for each state and exogenous factors, such as temperature and time of the day. The graphical illustration of CHSMM for the appliance-level modeling and forecasting is shown in Figure 1.8.



Figure 1.8 Graphical illustration of CHSMM

The appliance energy consumptions from period 1 to period k are denoted as  $\mathbf{Y} = \{y_1, y_2, \dots, y_k\}$ , and the latent states of the appliance are represented as  $X = \{x_1, x_2, ..., x_k\}$ . The state-space of the appliance is introduced to refer to the operating modes of the appliance. The state space is denoted as  $S = \{s_1, s_2, \dots, s_n\}$ , where *n* is the number of states. For example, A/C has two operating modes (on/off), and the state space of the A/C is {on, off}. In addition, the duration of each state D = $\{d_1, d_2, \dots, d_k\}$  and the exogenous variables  $\mathbf{Z} = \{z_1, z_2, \dots, z_k\}$  and  $\mathbf{W} = \{w_1^1, \dots, w_{d_k}^k\}$  are introduced in CHSMM structure shown in Figure 1.8. Duration space is introduced to refer to the possible duration for an appliance to stay in one operating mode, which is denoted as T = $\{t_1, t_2, ..., t_l\}$ , where l is the number of possible durations. CHSMM has a more complex relationship between states, durations of the state, observations, and exogenous variables in comparison with HMM. As can be seen in Figure 1.8, the current state  $x_k$  is conditioned on the previous state  $x_{k-1}$ , the previous duration of the state  $d_{k-1}$ , and exogenous variables  $z_k$ . The current state duration  $d_k$  is conditioned on the current state  $x_k$ , the previous state  $x_{k-1}$ , previous duration of the state  $d_{k-1}$ , and exogenous variables  $z_k$ . The observation  $y_{t_l}^k$  at time  $t_l$  within the period k is obtained by an emission function, which is formed by current state  $x_k$ , duration of the current state  $d_k$ , and exogenous variables  $w_{t_l}^k$ .

The mathematical expressions of CHSMM structure can be represented via (1)-(4).

$$\pi_i = \mathbb{P}(x_1 = s_i, d_1 = t_i) \tag{1}$$

$$A_{S}(z_{k}) = \mathbb{P}(x_{k}|x_{k-1}, d_{k-1}, z_{k})$$
<sup>(2)</sup>

$$A_D(z_k) = \mathbb{P}(d_k | x_k, x_{k-1}, d_{k-1}, z_k)$$
(3)

$$f(y_{t_l}^k|x_k, w_{t_l}^k) = \mathbb{P}(y_{t_l}^k|x_k, w_{t_l}^k)$$
(4)

where  $\pi_i$  is the initial distribution for the initial state  $x_1$  with the initial state duration  $d_1$  for CHSMM,  $A_S(z_k)$  is the state transition probability to the new state  $x_k$ ,  $A_D(z_k)$  is the duration transition probability to the new state duration  $d_k$ , and  $f(y_{t_l}^k|x_k, w_{t_l}^k)$  is the emission function to output the appliance consumption  $y_{t_l}^k$ 

CHSMM consists of three models, which are the state transition model, duration transition model, and emission function. In practice, the accuracy of the state transition model is usually high enough to forecast the next state. The emission function can be learned easily via a machine learning algorithm due to the energy consumption of an appliance following a fixed pattern within one operating mode. However, predicting the duration of the state is a challenging task because of the complex and unknown relationship between the duration of the state and the other factors. The duration transition model may undermine the performance of CHSMM if characteristics of the state duration of an appliance are not well captured.

The duration transition model of CHSMM in [8] is trained by MNLR, which is a classification method. However, there are potential training methods that can be used to enhance the performance of CHSMM. For example, A/C can have a very large duration space. The number of possible durations is huge, and the duration space of A/C is nearly continuous. In this case, using a regression-based method to train the duration transition model of CHSMM can be a better choice compared to the classification-based method since it is hard for classification-based methods to address the duration data characteristics with a huge number of classes. Moreover, a wrong prediction can result in a very large forecasting error by using classification-based methods for a nearly continuous dataset. In contrast, a wrong prediction of the state duration with regression-based methods can be still close to the actual duration of the state.

The refrigerator is the main appliance to illustrate the performance of CHSMM in [8]. A refrigerator in general runs in a cycle between compressor on and off. The duration of those two operating modes is about the same to keep the food at a desired cold temperature. The duration space of the refrigerator is small and limited, and the refrigerator has a relatively fixed running pattern. In this case, a classification-based method can be a good choice to achieve a higher accuracy of CHSMM.

The relationship between durations, states, and exogenous variables is highly nonlinear. The utilized regression-based method needs to have the ability to handle the nonlinearity. This work considers the regression Artificial Neural Network (ANN) to train the duration transition model of CHSMM. Due to the small number of features, a large and complex ANN is not necessary, instead, an ANN with a simple structure is enough. The state transition model of CHSMM is trained by MNRL as [8]. The emission function is trained via a simple ANN to deal with the nonlinear relationship between the appliance energy consumption and the other two features.

Appliance-level consumption data with the 1-min resolution is used in this work from the Pecan Street dataset [48]. One and a half month 1-min appliance consumption dataset (7/01/2018 to 8/15/2018) in the Austin area is used for the training and testing of CHSMM with different training methods, i.e., regression ANN and MNRL. The state of the appliance can be obtained by the load disaggregation model in the next subsection. The temperature and the time of the day information

are considered to be exogenous variables in CHSMM. The hourly temperature data in the Austin area for the same days (7/01/2018 to 8/15/2018) is obtained from Iowa Environmental Mesonet [49]. The obtained hourly temperature data is linearly interpolated to the 1-min resolution to match with the 1-min appliance-level consumption data. The appliance-level consumption forecasting using CHSMM is conducted every 15-min for over 6 hours. Refrigerators and A/C are two main appliances to test the impact of different training methods on the performance of CHSMM.

	MNLR RMSE (kW) NRMSE		Regression ANN	
			RMSE (kW)	NRMSE
A/C	2.43	0.66	1.41	0.38
Refrigerator	0.02	0.15	0.04	0.26

Table 1.2 Accuracy of the trained CHSMM for 6-hour ahead consumption prediction for	or one
appliance	

Table 1.2 shows the performance of the trained CHSMM for 6-hour ahead consumption prediction of A/C and refrigerator. As it can be seen in Table 1.2, using a regression ANN to train the duration transition model can improve the performance of CHSMM in terms of A/C consumption forecasting. The duration space of A/C is large, the number of possible durations is huge, and the duration space of A/C is nearly continuous. A regression-based training method can be a better choice to handle the data with such characteristics. In contrast, the classification-based method does not have a strong ability to deal with a large number of classes within the same training data. The accuracy of CHSMM with multinomial logistic regression (MNLR) is low for A/C consumption prediction. In addition, using a regression-based training method can avoid large forecasting errors. For example, a wrong prediction can still be close to the actual duration by using regression ANN. However, a classification-based training method may lead to a large forecasting error due to the characteristics of duration that is not well addressed. For refrigerator consumption forecasting, CHSMM with MNLR performs better than CHSMM with regression ANN. This is because the refrigerator has a relatively fixed running pattern and limited duration space. A classification-based method can be a good choice to train the duration transition probability to capture the characteristics of the duration of the refrigerator. The detailed comparisons of two types of training methods for CHSMM are shown in Figure 1.9 – Figure 1.12.



Figure 1.9 Performance of CHSMM for 6-hour ahead A/C consumption prediction with MNLR for the duration transition model



Figure 1.10 Performance of CHSMM for 6-hour ahead A/C consumption prediction with regression ANN for the duration transition model

Figure 1.9 and Figure 1.10 show the performance of CHSMM for 6-hour ahead A/C consumption prediction with different training methods for the duration transition model. The trained CHSMM with MNLR fails to track the consumption of A/C in Figure 1.9, while CHSMM with regression ANN is well tracking the A/C consumption in Figure 1.10. In Figure 1.9, the duration of the state is not accurately predicted, which means that the classification-based method does not capture the characteristics of the state duration for A/C. In addition, wrong predicted durations can result in large errors as shown in Figure 1.9. In contrast, the regression-based method is robust and can capture the characteristics of the duration of A/C. In Figure 1.10, the regression ANN can predict the duration of A/C. Moreover, the predicted consumption trajectory is not affected substantially by an incorrectly predicted duration in Figure 1.10.



Figure 1.11 Performance of CHSMM for 6-hour ahead refrigerator consumption prediction with MNLR for the duration transition model



Figure 1.12 Performance of CHSMM for 6-hour ahead refrigerator consumption prediction with regression ANN for the duration transition model

Figure 1.11 and Figure 1.12 show the comparison of the refrigerator consumption prediction with two different training methods for CHSMM. As it can be seen in Figure 1.11 and Figure 1.12, CHSMM with MNLR performs better than CHSMM with regression ANN for the refrigerator consumption prediction. The duration of the state for the refrigerator is well forecasted by MNLR in Figure 1.11. However, the regression ANN does not perform as well and results in some forecasting errors for predicting the refrigerator's duration as shown in Figure 1.12. By comparing Figure 1.11 and Figure 1.12, the classification-based methods can be a better choice to capture the duration characteristics of the appliance with limited duration space and nearly fixed running pattern.

	MNLR		Regression ANN	
	RMSE (kW)	NRMSE	RMSE (kW)	NRMSE
6 hours	1.64	0.56	1.30	0.45
7 days	1.68	0.58	1.38	0.47
14 days	1.77	0.59	1.50	0.49

Table 1.3 Averaged accuracy of the trained CHSMM for 10 A/Cs consumption forecasting

Table 1.4 Averaged accuracy of the trained CHSMM for 10 refrigerators consumption forecasting

	MNLR		Regression ANN	
	RMSE (kW)	NRMSE	RMSE (kW)	NRMSE
6 hours	0.08	0.34	0.10	0.37
7 days	0.08	0.18	0.09	0.19
14 days	0.08	0.16	0.09	0.17

To achieve a scalable comparison, a greater number of appliances and a longer length of forecasting period are considered. Table 1.3 and Table 1.4 show the averaged accuracy for 10 A/Cs and refrigerators consumption forecasting within the different lengths of the forecasting period. As it can be seen in Table 1.3, CHSMM with regression ANN can always achieve lower RMSE and NRMSE in comparison to CHSMM with MNLR for A/C consumption prediction. In contrast, CHSMM with MNLR performs better than CHSMM with regression ANN for refrigerator consumption prediction. Under different lengths of forecasting period and more appliances, Table 1.3 and Table 1.4 further confirm that the performance of CHSMM can be enhanced by selecting a proper training method based on the characteristics of the appliance.

Table 1.5 Averaged computational time for the training CHSMM

	MNLR (s)	Regression ANN (s)
A/C	3.1	3.4
Refrigerator	4.1	3.8

Training time is one of the important factors for appliance-level consumption modeling and forecasting in the residential area. Long training time is not preferred for an appliance-level forecasting model due to the large number of appliances in the residential area. Table 1.5 shows the average computational time comparison between two different training methods for CHSMM.

As it can be seen in Table 1.5, the training time of CHSMM is short for both methods. Moreover, the training time difference is small, which means that using different types of training methods to train CHSMM does not affect the training time a lot.

	Whole day       RMSE (kW)         NRMSE		Only 11:00 – 17:00	
			RMSE (kW)	NRMSE
A/C	1.41	0.38	1.40	0.38
Refrigerator	0.04	0.26	0.04	0.26

Table 1.6 Accuracy comparison using different portion of consumption data to the train CHSMM

Moreover, Table 1.6 evaluates the impact of using data from different time intervals on the accuracy of the trained CHSMM. As it can be seen, the accuracy difference between using data for the whole day and only 11:00 - 17:00 data for training is similar to each other. Note that the testing period is 11:00 - 17:00 for both cases. This means that if one only care about certain periods of the day, e.g., the period of a DR program, they can only use data from that period for training, which can reduce the size of needed storage memory.

#### 1.4.2 Non-intrusive appliance-level load monitoring

Appliance-level forecasting tool can help the utility improve their DSMS as shown in the previous subsection. The proposed forecasting algorithm needs historical appliance consumption data as input to predict the consumption of an appliance. However, the traditional smart meters can only provide the aggregated power consumption. One way to get the individual information of each appliance consumption is the non-intrusive load monitoring, which can decompose the aggregate power consumption, as provided by traditional smart meters, into its individual component appliances, without installing on-device monitoring equipment.

This work considers one load disaggregation model to achieve the non-intrusive appliance-level load monitoring. The objective function of the proposed load disaggregation model is minimizing two terms as shown in (5). The constraints (6) ensure that the appliance can only operate at one operating mode at time t. D(t) denotes as the total household demand measure from smart meter at time t.  $A_{(i,j)}$  denotes typical consumption level for appliance i at operating mode j.  $w_i$  denotes as weights for appliance i.  $\mu$  denotes penalty weight.  $x_{(i,j,t)}$  denotes as decision variable (binary) for appliance i at mode j at time t.

$$\min\left[\sum_{t=1}^{T} \left( D(t) - \sum_{i=1}^{N} \sum_{j=1}^{M} A_{(i,j)} x_{(i,j,t)} \right)^{2} \right] + \mu \sum_{i=1}^{N} \sum_{j=1}^{T} \sum_{t=2}^{T} w_{i} \left\| x_{(i,j,t-1)} - x_{(i,j,t)} \right\|$$
(5)

$$\sum_{j=1}^{M} x_{(i,j,t)} = 1$$
(6)

The first term minimizes the difference between measured demand and total appliances' consumption at typical operating mode. The second term is a regularization term penalizing the appliances that not always switch their operating mode. The second term tells the load

disaggregation model that the frequency of the appliance switching its operating mode. The weight  $w_i$  can be calculated based on the frequency of the operating mode transition such as  $\frac{1}{number of mode transition}$ . If the appliance generally has long duration at one mode (low frequency of mode transition, large  $w_i$ ), the term  $w_i ||x_{(i,j,t-1)} - x_{(i,j,t)}||_{\infty}$  would push the decision variable to be  $x_{(i,j,t-1)} = x_{(i,j,t)}$ . Since the second term pushes the decision variable to follow the frequency of the operating mode transition, the penalty weight  $\mu$  is needed to tell the model how much we should push.

μ	Computational time (s)	Accuracy refrigerator	Accuracy AC	Accuracy EV
0	3.4	0.716	0.983	0.978
10	49.8	0.716	0.988	0.985
100	51.8	0.718	0.997	1.000
1000	69.5	0.555	0.999	1.000

Table 1.7 Accuracy of the load disaggregation model

Pecan Street 1-min data is used for testing the model [39] in this analysis. Three appliances are considered to evaluate the performance of the load disaggregation model, i.e., refrigerator, AC, and EV. Different penalty weight  $\mu$  are tested. The model is programmed in Python and solved by Gurobi. As can be seen in Table 1.7, when the penalty weight  $\mu$  is 0, which means that there is no second term in the objective function, the computational time is around 3.4 sec. 71.6% of refrigerator's operating mode are correctly identified. 98.3% of AC's operating mode are correctly identified. 97.8% of EV's operating mode are correctly identified. It can also be observed that the computational time increases as the penalty weight  $\mu$  increases. In addition, as the penalty weight  $\mu$  increases, the accuracy of the load disaggregation model increases for identifying the operation model of AC and EV. However, when penalty weight  $\mu$  is set to be 1000. The accuracy of the model decreases to 55.5% for identifying the operating mode of the refrigerator.

The load disaggregation model performs well in terms of the appliances with larger consumption. The refrigerator has smaller consumption compared to the consumption of AC or EV. The load disaggregation model may treat the small refrigerator consumption as noise produced by AC and EV. This can be reason why the accuracy of the model for refrigerator is low.

#### 1.5 Conclusion

This work proposes an approach to model the load and DERs uncertainty and characterization. In addition, the load disaggregation model and CHSMM are considered to monitor and forecast the appliance-level consumption.

The simulation results of the proposed uncertainty modeling method show that the proposed approach can model and evaluate the uncertainty of the load and DERs via Gaussian distribution.

Moreover, the hypothesis of using Gaussian distribution has been checked by histogram and quantile-quantile plots, and Shapiro–Wilk test. In addition, exemplary figures show the hourly and 15-min uncertainty for aggregate prosumers. DR uncertainty can also be evaluated via the proposed approach.

Two different types of training methods are discussed in this work for the training of the duration transition model of CHSMM. The type of the training method needs to be carefully selected based on the characteristic of the appliance. CHSMM with regression-based training methods performs much better than CHSMM with classification-based training methods for the appliances with a large duration space and a huge number of possible durations. In contrast, classification-based training methods can be a better choice to achieve a higher accuracy of CHSMM for the appliances with a limited duration space and a nearly fixed running pattern. In addition, the training time of CHSMM is short and similar for both two types of training methods. Moreover, the prediction accuracy of models trained using data from similar prior time intervals is comparable with the prediction accuracy of models that use whole days of training mode of the appliance with larger consumption. However, the accuracy of the model is not high enough for identifying the appliance with low consumption, e.g., refrigerator.

## 2. Convex ACOPF for three-phase unbalanced system considering PV units with volt-var controller

#### 2.1 Introduction

The increasing penetration level of DERs is challenging the modern distribution network operation. IEEE Standards have been established to enhance power system operation and overcome challenges associated with integrating DERs in grid operation. SCC21 has developed IEEE Standard 1547-2018 for the interconnection and interoperability of DERs in distribution systems [9]. With the issued IEEE Standard 1547-2018, there is a need to update system operational tools to improve the representation of DERs in distribution operation. PV panels has been widely adopted in distribution systems. In addition, some adopted PV units have been installed with smart inverters that can adjust the power outputs of PV units and maintain the voltage within an acceptable range [9]. For example, excessive PV generation in distribution systems can lead to over-voltage issues, which can be mitigated via two options: (i) reactive power support under Q-V mode and (ii) active power curtailment of PV units. Thus, PV units with smart inverters constitute a set of dispatchable resources whose optimal operation can enhance the performance of distribution networks.

The assumption of DC optimal power flow typically made in transmission systems is not valid due to the high R/X ratio of distribution feeders. In addition, distribution networks are commonly unbalanced, and the mutual coupling between the three phases cannot be ignored. Therefore, a three-phase unbalanced AC optimal power flow (ACOPF) model is essential for distribution system operational scheduling. However, the three-phase unbalanced ACOPF is highly nonlinear and non-convex. Directly solving a nonlinear and non-convex model is not preferred because of the quality of the solution and the computational requirements.

Therefore, with the increasing penetration of DERs and the recommendations of IEEE standard 1547-2018, there is an urgent need to formulate a convex three-phase unbalanced ACOPF that incorporates the dispatching and performance of DERs in distribution grids.

#### 2.2 Literature review

Recently, some research efforts have been conducted to formulate the convexified three-phase unbalanced ACOPF model. The SDP-based three-phase unbalanced ACOPF model in [40] is proposed and solved by alternating direction method of multipliers. Reference [41] proposes a chordal relaxation-based SDP model for ACOPF in unbalanced systems considering DERs and voltage regulation transformers (VRTs) and provides a tighter convex model for VRTs to mitigate solution inexactness. Reference [42] convexifies nonlinear three-phase unbalanced ACOPF through a moment relaxation-based SDP model with a two-stage hierarchical algorithm to obtain the exact feasible solution. The SDP-based ACOPF for an unbalanced system in [43] accounts for the mixture of wye and delta-connection loads, DERs, and step voltage regulators.

The modeling of PV units in the distribution grid operation has been addressed in the literature. In [44]-[45], the active and reactive power and power factor limits of PV units are considered in optimal power flow (OPF). Reference [46] proposes three different control strategies with active

and reactive power limits for PV units to improve the voltage profile in distribution networks. However, the PV models in [44]-[46] may not satisfy IEEE standard 1547-2018 recommendations [9]. The DERs standard performance in IEEE standard 1547-2018 is studied in [47]. The authors of [47] formulate a decentralized approach to account for the standard characteristic in IEEE standard 1547-2018 for balanced systems. Reference [48] proposed a two-level Volt-VAr control scheme of PV units in which a 15-min dispatch and real-time adjustment are considered for PV units to enhance system operation. However, the unbalanced three-phase distribution grid characteristics are not well captured in [47]-[48]. To improve the modeling of DERs in system operational tools in unbalanced distribution grids, the project team incorporates the IEEE standard 1547-2018 characteristic of PV units with two different convex three-phase unbalanced ACOPF models.

## 2.3 Convex ACOPF for three-phase unbalanced system considering PV units with volt-var controller

The project team has proposed two convex ACOPF models for DERs scheduling in the unbalanced network: a convex second order cone programming (SOCP)-based ACOPF model, and a convex current-voltage (IV)-based ACOPF model.

#### 2.3.1 Convex ACOPF for three-phase unbalanced system: Two-stage Convex SOCP-based ACOPF Model

In the three-phase unbalanced network, the complex power flow on the line  $S_{ij\phi}^{L}$  from bus *i* to bus *j* at phase  $\phi$  can be calculated through the voltage  $V_{i\phi}$  multiplied by the conjugate of the line current (phasors are in bold font). The formulation of power flow of a three-phase line is shown in (7).

$$\boldsymbol{S}_{ij\phi}^{L} = \boldsymbol{V}_{i\phi} \left( \sum_{\phi' \in \Phi} \left( \boldsymbol{V}_{i\phi'} - \boldsymbol{V}_{j\phi'} \right) \boldsymbol{Y}_{ij}^{\phi\phi'} \right)^{*}$$
(7)

where  $V_{i\phi}$  is the voltage at bus *i* on phase  $\phi$ ;  $Y_{ij}^{\phi\phi'}$  is the admittance of the path from bus *i* phase  $\phi$  to bus *j* phase  $\phi'$ ; note that  $Y_{ij}^{\phi\phi}$  indicates self-impedance of phase  $\phi$  and  $Y_{ij}^{\phi\phi'}$  ( $\phi \neq \phi'$ ) indicates mutual impedances between phases  $\phi$  and  $\phi'$  with phasor  $V_{i\phi} = V_{i\phi} \angle \theta_{i\phi}$  and  $Y_{ij}^{\phi\phi'} = g_{ij}^{\phi\phi'} + j b_{ij}^{\phi\phi'}$ .  $V_{i\phi}$  and  $\theta_{i\phi}$  are the voltage magnitude and the voltage angle at bus *i* phase  $\phi$  respectively. By substituting  $V_{i\phi}$  and  $Y_{ij}^{\phi\phi'}$  in (7),  $S_{ij\phi}^{L}$  can be written as (8).

$$S_{ij\phi}^{L} = V_{i\phi}V_{i\phi}\left(g_{ij}^{\phi\phi} - jb_{ij}^{\phi\phi}\right) + \sum_{\phi'\in\Phi\setminus\phi}V_{i\phi}V_{i\phi'}\left(\cos\theta_{ii}^{\phi\phi'} + j\sin\theta_{ii}^{\phi\phi'}\right)\left(g_{ij}^{\phi\phi'} - jb_{ij}^{\phi\phi'}\right) - \sum_{\phi'\in\Phi}\left(\cos\theta_{ij}^{\phi\phi'} + j\sin\theta_{ij}^{\phi\phi'}\right)V_{i\phi'}V_{j\phi'}\left(g_{ij}^{\phi\phi'} - jb_{ij}^{\phi\phi'}\right)$$
(8)

Three auxiliary variables (9)-(11) are introduced to reformulate the nonlinear three-phase unbalanced formulation (8) and eventually convexify it in (12)-(14) and (16).

$$c_{ij}^{\phi\phi'} = V_{i\phi}V_{j\phi'}\cos\theta_{ij}^{\phi\phi'} \tag{9}$$

$$e_{ij}^{\phi\phi'} = V_{i\phi}V_{j\phi'}\sin\theta_{ij}^{\phi\phi'} \tag{10}$$

$$u_{i\phi} = \left(V_{i\phi}\right)^2 \tag{11}$$

By substituting three auxiliary variables into (8), the active and reactive power flows on the line from bus *i* to bus *j* on phase  $\phi$  can be reformulated as a convex form (12)-(13). The relationship of the auxiliary variables  $c_{ij}^{\phi\phi'}$  and  $s_{ij}^{\phi\phi'}$  are presented via (14)-(15).

$$P_{ij\phi}^{L} = g_{ij}^{\phi\phi} u_{i\phi} + \sum_{\phi' \in \Phi \setminus \phi} \left( g_{ij}^{\phi\phi'} c_{ii}^{\phi\phi'} + b_{ij}^{\phi\phi'} e_{ii}^{\phi\phi'} \right) - \sum_{\phi' \in \Phi} \left( g_{ij}^{\phi\phi'} c_{ij}^{\phi\phi'} + b_{ij}^{\phi\phi'} e_{ij}^{\phi\phi'} \right),$$

$$\forall ij\phi \in L$$

$$Q_{ij\phi}^{L} = -b_{ij}^{\phi\phi} u_{i\phi} + \sum_{\phi' \in \Phi \setminus \phi} \left( g_{ij}^{\phi\phi'} e_{ii}^{\phi\phi'} - b_{ij}^{\phi\phi'} c_{ii}^{\phi\phi'} \right) - \sum_{\phi' \in \Phi} \left( g_{ij}^{\phi\phi'} e_{ij}^{\phi\phi'} - b_{ij}^{\phi\phi'} c_{ij}^{\phi\phi'} \right),$$

$$\forall ij\phi \in L$$

$$(12)$$

$$c_{ij}^{\phi\phi'} = c_{ji}^{\phi'\phi}, \ e_{ij}^{\phi\phi'} = -e_{ji}^{\phi'\phi}$$
(14)

$$\left(c_{ij}^{\phi\phi'}\right)^2 + \left(e_{ij}^{\phi\phi'}\right)^2 = u_{i\phi}u_{j\phi'} \tag{15}$$

where  $P_{ij\phi}^{L}$  and  $Q_{ij\phi}^{L}$  are active and reactive power flow on the line from bus *i* to bus *j* at phase  $\phi$ . The constraint (15) shows the exact relationship between auxiliary variables; however, it introduces non-convexity in the model. Therefore, the non-convex constraint (15) is relaxed as a convex SOC constraint in (16).

$$\left(c_{ij}^{\phi\phi'}\right)^2 + \left(e_{ij}^{\phi\phi'}\right)^2 \le u_{i\phi}u_{j\phi'} \tag{16}$$

A general three-phase unbalanced ACOPF can be formulated as a convex SOCP-based optimization problem as shown in (17)-(24) with the core constraints (12)-(14), (16). In this general model, the PV units without VVC are non-controllable. As a result, these PV units inject maximum available active power with zero reactive power into the distribution grid.

$$\min \sum_{\forall g \in G} \rho^G P_g^G + \sum_{\forall h \in H} \rho^{pv} \left( P_h^{pv} - P_{i\phi, if i\phi \in \mathcal{D}(h)}^D \right)$$
(17)

$$\sum_{\forall g \in G(i\phi)} P_g^G + \sum_{\forall h \in H(i\phi)} P_h^{p\nu} = \sum_{\forall i\phi \in D} P_{i\phi}^D + \sum_{\forall j \in B(i\phi)} P_{ij\phi}^L, \ \forall i\phi \in B$$
(18)

$$\sum_{\forall g \in G(i\phi)} Q_g^G + \sum_{\forall h \in H(i\phi)} Q_h^{pv} = \sum_{\forall i\phi \in D} Q_{i\phi}^D + \sum_{\forall j \in B(i\phi)} Q_{ij\phi}^L, \ \forall i\phi \in B$$
(19)

$$P_h^{pv} = P_{pv,h}^{max}, \ \forall h \in H$$
(20)

$$Q_h^{pv} = 0, \ \forall h \in H \tag{21}$$

$$\left(V_{i\phi}^{L}\right)^{2} \leq u_{i\phi} \leq \left(V_{i\phi}^{U}\right)^{2}, \ \forall i\phi \in B \backslash G$$

$$\tag{22}$$

$$u_g = \left(V_g\right)^2, \ \forall g \in G \tag{23}$$

$$\theta_g = \theta_g^{sub}, \ \forall g \in G \tag{24}$$

where  $\rho^{G}$  and  $\rho^{pv}$  are energy prices from bulk system and PV units.  $P_{g}^{G}$  and  $P_{h}^{pv}$  are the active power from the substation and PV unit.  $Q_{g}^{G}$  is the reactive power from the substation.  $P_{i\phi}^{D}$  and  $Q_{i\phi}^{D}$ 

are the active and reactive power demand.  $P_{pv,h}^{max}$  is the maximum power point of the PV unit.  $V_{i\phi}^{U}$  and  $V_{i\phi}^{L}$  are the upper and lower voltage limits.  $\theta_{g}$  is the voltage angle.

The objective function (17) minimizes the total system operating cost including the cost of energy purchased from the upstream wholesale market and the cost of solar generation surplus paid to PV owners. The active and reactive power balances at each node are constrained by (18)-(19). The VVC enables changing the active and reactive power output of PV units and provide voltage regulation service. For the PV units without VVC, the PV units' active and reactive outputs are modeled as (20)-(21). Voltage limits of nodes are given by (22). Constraints (23)-(24) fix voltage magnitude and angle for nodes connected to the substation.

However, the reformed formulations (12)-(14), (16) provide no guarantee of a correct and exact solution. One reason is that the mutual impedance of the three-phase line creates the virtual loops, which makes the network non-radial and complicates the determination of an exact solution using the SOCP-based formulations (12)-(14), (16).

This work proposes new bounding constraints for auxiliary variables  $c_{ij}^{\phi\phi'}$  and  $e_{ij}^{\phi\phi'}$  to narrow down the feasible region, address the problems of virtual paths created by mutual impedances, and bound the terms associated with different phases via Taylor series approximation. The formulations (25)-(26) are linearized expressions of auxiliary variables  $c_{ij}^{\phi\phi'}$  and  $e_{ij}^{\phi\phi'}$  in (9)-(10) by applying the first-order Taylor series approximation.

$$c_{ij}^{\phi\phi'} = \sqrt{u_{i\phi}^{(t)}u_{j\phi'}^{(t)}} \left(\theta_{i\phi}^{(t)} - \theta_{j\phi'}^{(t)}\right) \sin\left(\theta_{i\phi}^{(t)} - \theta_{j\phi'}^{(t)}\right) + \frac{\sqrt{u_{i\phi}^{(t)}}}{2\sqrt{u_{i\phi}^{(t)}}} \cos\left(\theta_{i\phi}^{(t)} - \theta_{j\phi'}^{(t)}\right) u_{i\phi} + \frac{\sqrt{u_{i\phi}^{(t)}}}{2\sqrt{u_{j\phi'}^{(t)}}} \cos\left(\theta_{i\phi}^{(t)} - \theta_{j\phi'}^{(t)}\right) u_{j\phi'} - \sqrt{u_{i\phi}^{(t)}u_{j\phi'}^{(t)}} \sin\left(\theta_{i\phi}^{(t)} - \theta_{j\phi'}^{(t)}\right) (\theta_{i\phi} - \theta_{j\phi'})$$
(25)
$$e_{ij}^{\phi\phi'} = -\sqrt{u_{i\phi}^{(t)}u_{j\phi'}^{(t)}} \left(\theta_{i\phi}^{(t)} - \theta_{j\phi'}^{(t)}\right) \cos\left(\theta_{i\phi}^{(t)} - \theta_{j\phi'}^{(t)}\right) + \frac{\sqrt{u_{i\phi}^{(t)}}}{2\sqrt{u_{i\phi}^{(t)}}} \sin\left(\theta_{i\phi}^{(t)} - \theta_{j\phi'}^{(t)}\right) u_{i\phi} + \frac{\sqrt{u_{i\phi}^{(t)}}}{2\sqrt{u_{i\phi}^{(t)}}} \sin\left(\theta_{i\phi}^{(t)} - \theta_{i\phi}^{(t)}\right) u_{i\phi} + \frac{\sqrt{u_{i\phi}^{(t)}}}{2\sqrt{u_{i\phi}^{(t)}}} \sin\left(\theta_{i\phi}^{(t)} - \theta_{i\phi}^{(t)}\right) u_{i\phi} + \frac{\sqrt{u_{i\phi}^{(t)}}}{2\sqrt{u_{i\phi}^{(t)}}} \sin\left(\theta_{i\phi}^{(t)} - \theta_{i\phi}^{(t)}\right) u_{i\phi} + \frac{\sqrt{u_{i\phi}^{(t)}}}{2\sqrt{u_{i\phi}^{(t)}}} \sin\left(\theta_{i\phi}^{(t$$

$$\frac{\sqrt{u_{i\phi}^{(t)}}}{2\sqrt{u_{j\phi'}^{(t)}}}\sin\left(\theta_{i\phi}^{(t)} - \theta_{j\phi'}^{(t)}\right)u_{j\phi'} + \sqrt{u_{i\phi}^{(t)}u_{j\phi'}^{(t)}}\cos\left(\theta_{i\phi}^{(t)} - \theta_{j\phi'}^{(t)}\right)(\theta_{i\phi} - \theta_{j\phi'})$$
(26)

Since Taylor series first-order expansion base points (e.g.,  $u_{i\phi}^{(t)}$  and  $\theta_{i\phi}^{(t)}$ ) are introduced in (25)-(26), those base points need to be iteratively updated in the SOCP-based ACOPF model to obtain an exact and feasible solution. In this regard, a two-stage algorithm is developed to solve the proposed three-phase unbalanced SOCP-based ACOPF model with new bounding constraints (25)-(26).

A two-stage algorithm is proposed to solve the convex SOCP-based ACOPF. The convex ACOPF is solved in Stage 1.



Figure 2.1 Proposed two-stage algorithm for three-phase unbalanced SOCP-based ACOPF

The overall two-stage algorithm is presented in Figure 2.1. The algorithm updates the Taylor series base points and adds SOC constraints if needed. Stage 1 starts from an initial point, which can be a flat start for all nodes with voltage magnitudes of the substation and voltage angles 0°, -120°,  $+120^{\circ}$  for phases a, b and c, respectively. Moreover, a power flow solution of voltage magnitudes and angles from a similar system condition can be used as an initial point. With the selected initial point, the initial Taylor series base points (i.e.,  $u_{i\phi}^{(t)}$  and  $\theta_{i\phi}^{(t)}$ ) can be calculated for the linearized constraints (25)-(26). Then, Stage 1 goes into the loop to make the voltage magnitudes and angles converged. In the first iteration of the loop, the initial Taylor series base points are used to update linearized constraints (25)-(26) and the convex SOCP-based ACOPF (12)-(14), (17)-(24) is solved. After obtaining the solution of the convex ACOPF model (12)-(14), (17)-(24), new Taylor series base points are calculated based on the obtained solution to prepare for the update in the next iteration. If the updated voltage magnitudes and angles are close enough to the previous values between iterations, the update of the Taylor series base points has converged. In that case, the algorithm can proceed to Stage 2. Otherwise, Stage 1 continues to update Taylor series base points in the loop until they meet the tolerance criterion. This work considers tolerance criterion to be  $1 \times 10^{-3}$  p.u. in the case study. In Stage 2, the absolute SOC relaxation errors  $u_{i\phi}u_{j\phi'} - (c_{ij}^{\phi\phi'})^2 - (c_{ij}^{\phi\phi'})^2$
$\left(e_{ij}^{\phi\phi'}\right)^2$  are checked. If the absolute relaxation errors are minor, i.e., less than 2×10<sup>-6</sup> p.u., the algorithm stops and provides the obtained solution. Otherwise, the SOC constraint (16) is added, and the algorithm goes back to Stage 1.

# 2.3.2 Convex ACOPF for three-phase unbalanced system: Convex IV-based ACOPF Model

An unbalanced distribution network with a graph representation of  $\mathcal{G} = (\mathcal{J}, \mathcal{H})$  is considered, where  $\mathcal{J}$  is a set of buses and  $\mathcal{H}$  is a set of distribution lines. Every bus and line in  $\mathcal{G}$  can have three phases, i.e., set  $\psi = \{a, b, c\}$ . In the proposed formulation, the inverse of x is represented by  $(x)^{-1}$ . The voltage difference over a three-phase distribution line  $\mathcal{L} \in \mathcal{H}$  in phase  $\phi$  is formulated based on the self and mutual impedances and admittances of the line as follows:

$$V_n^{\phi} - V_m^{\phi} = Z_{\mathcal{L}}^{\phi,\phi} I_{\mathcal{L}}^{\phi} + \sum_{p \in \psi, p \neq \phi} Z_{\mathcal{L}}^{\phi,p} I_{\mathcal{L}}^p - \frac{1}{2} \sum_{p \in \psi} Z_{\mathcal{L}}^{\phi,p} \left( \sum_{k \in \psi} y_{\mathcal{L}}^{p,k} V_n^k \right), \forall \phi \in \psi, \mathcal{L} \in \mathcal{H}$$
(27)

where line  $\mathcal{L}$  connects bus *n* to bus *m*. The current from bus *n* to bus *m* in the phase  $\phi$  of a distribution line is defined based on (28) as follows:

$$I_{\ell}^{\phi} = \left(Z_{\ell}^{\phi,\phi}\right)^{-1} \left[V_{n}^{\phi} - V_{m}^{\phi} - \sum_{p \in \psi, p \neq \phi} Z_{\ell}^{\phi,p} \left(I_{\ell}^{p}\right) + \frac{1}{2} \sum_{p \in \psi} Z_{\ell}^{\phi,p} \left(\sum_{k \in \psi} y_{\ell}^{p,k} V_{n}^{k}\right)\right], \forall \phi \in \psi, \mathcal{L} \in \mathcal{H}$$

$$(28)$$

As illustrated in (28), the current in the three-phase distribution line  $\mathcal{L}$  in the phase  $\phi$  is obtained not only based on voltage, impedance, and admittance values of the phase  $\phi$  but also on the voltage of the other phases as a result of the mutual impedances and admittances between the phase  $\phi$  and the other two phases of the line. The real and imaginary parts of the current in phase  $\phi$  of line  $\mathcal{L} \in$  $\mathcal{H}$  (i.e.,  $I_{\mathcal{L}}^{\phi}$  in (28)) are given in (29)-(30), respectively.

$$I_{\ell}^{r,\phi} = \left(R_{\ell}^{\phi,\phi}\right)^{-1} \left[V_{n}^{r,\phi} - V_{m}^{r,\phi} - \sum_{p \in \psi, p \neq \phi} R_{\ell}^{\phi,p} I_{\ell}^{r,p} - \frac{1}{2} \sum_{p \in \psi} R_{\ell}^{\phi,p} \left(\sum_{k \in \psi} y_{\ell}^{p,k} V_{n}^{im,k}\right) + \sum_{p \in \psi} X_{\ell}^{\phi,p} \left(I_{\ell}^{im,p} - \frac{1}{2} \sum_{k \in \psi} y_{\ell}^{p,k} V_{n}^{r,k}\right)\right], \forall \phi \in \psi, \mathcal{L} \in \mathcal{H}$$

$$(29)$$

$$I_{\ell}^{im,\phi} = \left(R_{\ell}^{\phi,\phi}\right)^{-1} \left[V_{n}^{im,\phi} - V_{m}^{im,\phi} - \sum_{p \in \psi, p \neq \phi} R_{\ell}^{\phi,p} I_{\ell}^{im,p} + \frac{1}{2} \sum_{p \in \psi} R_{\ell}^{\phi,p} \left(\sum_{k \in \psi} y_{\ell}^{p,k} V_{n}^{r,k}\right) - \sum_{p \in \psi} X_{\ell}^{\phi,p} \left(I_{\ell}^{r,p} + \frac{1}{2} \sum_{k \in \psi} y_{\ell}^{p,k} V_{n}^{im,k}\right)\right], \forall \phi \in \psi, \mathcal{L} \in \mathcal{H}$$

$$(30)$$

The components of the injected current in phase  $\phi$  of bus *n* is shown in (31)-(32), where  $\delta(n)$  is set of lines connected to bus *n*.

$$I_n^{r,\phi} = \sum_{\mathcal{L} \in \delta(n)} I_{\mathcal{L}}^{r,\phi} , \forall \phi \in \psi, n \in \mathcal{J}$$
(31)

$$I_n^{im,\phi} = \sum_{\mathcal{L}\in\mathcal{S}(n)} I_{\mathcal{L}}^{im,\phi} , \forall \phi \in \psi, n \in \mathcal{J}$$
(32)

DERs, e.g., solar panels, are connected to the system via distribution transformers. For phase  $\phi$  of bus  $n \in \mathcal{J}$ , the power balance formulations are given in (33)-(34).

$$\sum_{\forall g \in g(n)} P_{g,\phi}^{G} - \sum_{\forall l \in l(n)} d_{p,l,\phi} - \sum_{\forall r \in R(n)} NL_{r,\phi}^{TR} = V_{n}^{r,\phi} I_{n}^{r,\phi} + V_{n}^{im,\phi} I_{n}^{im,\phi}, \forall \phi \in \psi, n \in \mathcal{J}$$

$$(33)$$

$$\sum_{\forall g \in g(n)} Q_{g,\phi}^G + \sum_{\forall c \in C(n)} Q_{c,\phi}^C - \sum_{\forall l \in l(n)} d_{q,l,\phi} = V_i^{im,a} I_i^{r,a} - V_i^{r,a} I_i^{im,a}, \forall \phi \in \psi, n \in \mathcal{J}$$

$$(34)$$

where  $NL_{r,\phi}^{TR}$ ,  $P_{g,\phi}^{G}$ ,  $Q_{g,\phi}^{G}$ ,  $d_{p,l,\phi}$ ,  $d_{q,l,\phi}$ , and  $Q_{c,\phi}^{C}$  are no load loss of transformer r at phase  $\phi$ , active and reactive power of DER g at phase  $\phi$ , active and reactive power of load l at phase  $\phi$ , and reactive power of capacitor unit c at phase  $\phi$ , respectively.

The power balance equations (33)-(34) are nonlinear due to the products of current and voltage variables. An iterative first-order approximation of the Taylor expansion has been utilized to develop the linear model of (35)-(36) as follows:

$$\underline{\sum_{\forall g \in g(n)} P_{g,\phi}^{G} - \sum_{\forall l \in l(n)} d_{p,l,\phi} - \sum_{\forall r \in R(n)} NL_{r,\phi}^{TR} = \underline{V_{n,it-1}^{r,\phi}} I_{n}^{r,\phi} + \underline{V_{n,it-1}^{im,\phi}} I_{n}^{im,\phi} + \underline{I_{n,it-1}^{r,\phi}} V_{n}^{r,\phi} + \underline{I_{n,it-1}^{im,\phi}} I_{n,it-1}^{r,\phi} - \underline{V_{n,it-1}^{im,\phi}} I_{n,it-1}^{im,\phi}, \forall \phi \in \psi, n \in \mathcal{J}$$
(35)

$$\underline{\sum_{\forall g \in g(n)} Q_{g,\phi}^{G} + \sum_{\forall c \in C(n)} Q_{c,\phi}^{C} - \sum_{\forall l \in l(n)} d_{q,l,\phi} = \underline{V_{n,it-1}^{im,\phi}} I_{n}^{r,\phi} - \underline{V_{n,it-1}^{r,\phi}} I_{n}^{im,\phi} + \underline{I_{n,it-1}^{r,\phi}} V_{n}^{im,\phi} - \underline{I_{n,it-1}^{im,\phi}} V_{n}^{r,\phi} - \underline{V_{n,it-1}^{im,\phi}} I_{n,it-1}^{r,\phi} + \underline{V_{n,it-1}^{r,\phi}} I_{n,it-1}^{im,\phi}, \forall \phi \in \psi, n \in \mathcal{J}$$
(36)

where  $\underline{V_{n,it-1}^{r,\phi}}$ ,  $\underline{V_{n,it-1}^{im,\phi}}$ ,  $\underline{I_{n,it-1}^{r,\phi}}$  and  $\underline{I_{n,it-1}^{im,\phi}}$  are outputs of the previous iteration (i.e., it - 1) and are considered as input parameters in the current iteration. These linearized formulations are also utilized here. The voltage magnitude constraints for bus *n* at phase  $\phi$  is given in (37)-(38).

$$0.95 \le V_n^{\phi} \le 1.05, \forall \phi \in \psi, n \in \mathcal{J}$$
(37)

$$V_n^{\phi} = \sqrt{V_n^{r,\phi^2} + V_n^{im,\phi^2}}, \forall \phi \in \psi, n \in \mathcal{J}$$
(38)

Constraint (38) is linearized as shown in (39).

$$V_{n}^{\phi} = \frac{\underline{v_{n,it-1}^{r,\phi}}}{\sqrt{\underline{v_{n,it-1}^{r,\phi}}^{2} + \underline{v_{n,it-1}^{im,\phi}}^{2}}} V_{n}^{r,\phi} + \frac{\underline{v_{n,it-1}^{im,\phi}}}{\sqrt{\underline{v_{n,it-1}^{r,\phi}}^{2} + \underline{v_{n,it-1}^{im,\phi}}^{2}}} V_{n}^{im,\phi}, \forall \phi \in \psi, n \in \mathcal{J}$$
(39)

The proposed accurate IV-based ACOPF can be used for different distribution system studies by considering different objective functions, e.g., minimizing operation cost, minimizing measurements error. In most of the prior distribution ACOPF models, such as the DistFlow model, line flow constraints are nonlinear, which are linearized by ignoring line losses in the distribution systems. In the proposed ACOPF model, not only are line losses modeled but also the line flow constraints are inherently linear.

#### 2.3.3 Q-V curve modeling of PV units with VVC in convex ACOPF

The project team has developed formulations to characterize Q-V curve (from IEEE 1547-2018 standard; Figure 2.2) of the PV units with VVC in the ACOPF. The constraints representing this Q-V characteristic are shown in (40)-(50), where  $Q^{max} = Q_1 = -Q_4$  and  $V_1 - V_4$  are represented using  $U_1 - U_4$ , i.e., the equivalent auxiliary form, in the proposed SOCP-based ACOPF model.



Figure 2.2 Q-V characteristic of PV units with VVC

The project team has proposed two different models to incorporate Q-V characteristic of PV units having VVC in the proposed two ACOPF models: (A) default setpoints from IEEE 1547-2018 standard and (B) co-optimized setpoints based on allowable range of IEEE 1547-2018 standard.

The Q-V characteristic of PV units having VVC with  $U_1 - U_4$  and  $Q^{max}$  to be default setpoints are modeled as (40)-(50). Note that the binary variable  $z_1, z_2, z_3, z_4, z_5$  are defined for each operating zone.

• First if-then equation (zone 1):

$$U_{i,x} \le U_1 + M z_1 \tag{40}$$

$$-Mz_{1} \leq Q_{i,x}^{PV} - Q_{i,x}^{PV,max} \leq Mz_{1}$$
(41)

Second if-then equation (zone 2):

$$-Mz_2 + U_1 \le U_{i,x} \le U_2 + Mz_2 \tag{42}$$

$$-Mz_{2} \leq Q_{i,x}^{PV} - \frac{Q_{i,x}^{PV,max}}{U_{2} - U_{1}} \left( U_{2} - U_{i,x} \right) \leq Mz_{2}$$

$$\tag{43}$$

• Third if-then equation (zone 3):

$$-Mz_3 + U_2 \le U_{i,x} \le U_3 + Mz_3 \tag{44}$$

$$-Mz_3 \le Q_{i,x}^{PV} \le Mz_3 \tag{45}$$

• Fourth if-then equation (zone 4):

$$-Mz_4 + U_3 \le U_{i,x} \le U_4 + Mz_4 \tag{46}$$

$$-Mz_4 \le Q_{i,x}^{PV} - \frac{Q_{i,x}^{PV,max}}{U_3 - U_4} \left( U_{i,x} - U_3 \right) \le Mz_4$$
(47)

• Fifth if-then equation (zone 5):

$$-Mz_5 + U_4 \le U_{i,x} \tag{48}$$

$$-Mz_5 \le Q_{i,x}^{PV} + Q_{i,x}^{PV,max} \le Mz_5 \tag{49}$$

Mutually exclusive constraint for if-then constraints

$$z_1 + z_2 + z_3 + z_4 + z_5 \le 4 \tag{50}$$

Also, for PV units with VVC, constraints (51)-(58) are added to the model. Constraints (55)-(58) are defined based on IEEE 1547 standard where  $V_{ref}$  considered to be 1.

$$Q_{n,\phi}^{PV^2} + P_{n,\phi}^{PV^2} \le \left(\frac{P_{n,\phi}^{PV,max}}{PF_{n,\phi}^{min}}\right)^2 \tag{51}$$

$$P_{\nu,\phi}^{PV} \le P_{n,\phi}^{PV,a\nu a} \tag{52}$$

$$0 \le Q_{p\nu,n,\phi}^{max} < \frac{P_{n,\phi}^{PV,max}}{PF_{n,\phi}^{min}}$$
(53)

$$-Q_{p\nu,n,\phi}^{max} \le Q_{n,\phi}^{PV} < Q_{p\nu,n,\phi}^{max}$$
(54)

$$V_{ref} - 0.18 \le V_{1,n,\phi} < V_{2,n,\phi} - 0.02 \tag{55}$$

$$V_{ref} - 0.03 \le V_{2,n,\phi} < V_{ref} \tag{56}$$

$$V_{ref} \le V_{3,n,\phi} < V_{ref} + 0.03 \tag{57}$$

$$V_{3,n,\phi} + 0.02 \le V_{4,n,\phi} < V_{ref} + 0.18 \tag{58}$$

The project team has extended the proposed IV-based and SOCP-based ACOPF models to identify the setpoints  $(U_1 - U_4 \text{ and } Q^{max})$  of VVC in the Q-V characteristic of PV unit (Figure 2.2) within the allowable range of IEEE 1547-2018 standard. Co-optimizing the setpoints of VVC results in non-linear formulations in (43) and (47). Taylor expansion is introduced to linearize these formulations (43) and (47) as (59) and (60), respectively.

$$-Mz_{2} \leq Q_{i,x}^{PV} - \frac{U_{2,ix}^{t} - U_{i,x}^{t}}{U_{2,ix}^{t} - U_{1,ix}^{t}} Q_{pv,ix}^{max} + \frac{Q_{pv,ix}^{max,t}}{U_{2,ix}^{t} - U_{1,ix}^{t}} U_{i,x} - \frac{Q_{pv,ix}^{max,t}(U_{2,ix}^{t} - U_{i,x}^{t})}{(U_{2,ix}^{t} - U_{1,ix}^{t})^{2}} U_{1,ix} + \frac{Q_{pv,ix}^{max,t}(U_{1,ix}^{t} - U_{1,ix}^{t})^{2}}{(U_{2,ix}^{t} - U_{1,ix}^{t})^{2}} U_{2,ix} \leq Mz_{2}$$

$$-Mz_{4} \leq Q_{i,x}^{PV} - \frac{(U_{i,x}^{t} - U_{3,ix}^{t})}{U_{3,ix}^{t} - U_{4,ix}^{t}} Q_{pv,ix}^{max} - \frac{Q_{pv,ix}^{max,t}}{U_{3,ix}^{t} - U_{4,ix}^{t}} U_{i,x} + \frac{Q_{pv,ix}^{max,t}(U_{i,x}^{t} - U_{4,ix}^{t})}{(U_{3,ix}^{t} - U_{4,ix}^{t})^{2}} U_{3,ix} - \frac{Q_{pv,ix}^{max,t}}{U_{3,ix}^{t} - U_{4,ix}^{t}} U_{i,x} + \frac{Q_{pv,ix}^{max,t}(U_{i,x}^{t} - U_{4,ix}^{t})}{(U_{3,ix}^{t} - U_{4,ix}^{t})^{2}} U_{3,ix} - \frac{Q_{pv,ix}^{max,t}(U_{i,x}^{t} - U_{4,ix}^{t})}{(U_{3,ix}^{t} - U_{4,ix}^{t})^{2}} U_{4,ix} \leq Mz_{4}$$

$$(60)$$

# 2.3.4 Result: Q-V curve modeling of PV units with Volt-Var controller in IV-based ACOPF:

The proposed IV-based ACOPF formulation is a convex model, which minimizes operation cost. The proposed model is tested on both primary and secondary feeder of the utility partner feeder with 10588 buses, 7943 lines, 742 nodes with PV units, 1737 load nodes. The proposed model is tested on March 15, 2019, 2 pm with overvoltage issues. In order to evaluate performance of the model with considering shunt elements of distribution lines, 110 nodes equipped with PV units with VVC.

The obtained co-optimized setpoints of PV units with VVC are validated using OpenDSS software. Voltage magnitude of all buses in OpenDSS based on co-optimized setpoints of PV units with VVC is compared with the results of IV-based ACOPF in Figure 2.3. It can be observed that voltage magnitude of all nodes obtained by the proposed IV-based ACOPF matches with OpenDSS, which implies that the proposed IV-based ACOPF approach models an unbalanced distribution system with mutual impedance, shunt elements, and detailed model of DERs and transformers accurately.



Figure 2.3 Voltage magnitude comparison of all buses in OpenDSS and IV-based ACOPF based on co-optimized setting of PV units with VVC.

Also, the results of the IV-based ACOPF based on default setting and co-optimized setting of Q-V curve are compared with different numbers of PV units with VVC. In the default setting, set points of PV units with VVC are considered as parameter based on IEEE 1547 standard. In this regard,  $V_{1,n,\phi}$ ,  $V_{2,n,\phi}$ ,  $V_{3,n,\phi}$ ,  $V_{4,n,\phi}$  are considered to be 0.94, 0.98, 1.02, and 1.06.  $Q_{pv,n,\phi}^{max}$  is defined based on rating of the PV units. Table 1.8 and Table 1.9 show the comparison of two models.

Table 2.1 Comparison	of the active power	curtailment and	cost saving	of default	and co-
	optimized settings	of PV units with	VVC.		

NO. of		Default	setpoints	Co-optimiz	ed setpoints	
nodes with PV and VVC	∑ Pmax of PV (kW)	∑ active power curtailment (kW)	% active power curtailment	∑ active power curtailment (kW)	% active power curtailment	Cost saving (%)
85	589.05	77	13.07	6.83	1.16	5.32
80	562.2	242.25	43.09	64.3	11.44	15.42
75	534.4	272	50.90	67.93	12.71	18.17
70	506.85	308.35	60.84	78.55	15.50	21.11

Table 2.2 Comparison of the reactive power output and Qmax of default and co-optimized settings of PV units with VVC.

NO. of nodes	Default setpoints		Co-optimized setpoints		
VVC	∑ reactive power of PV (kVAR)	$\sum$ Qmax of PV (kVAR)	∑ reactive power of PV (kVAR)	$\sum_{kVAR} Qmax of PV$	
85	-313.8	468.8	-417.95	565.90	
80	-276.2	447.65	-495.75	575.95	
75	-259.2	426.1	-487.65	538.85	
70	-242	404.2	-471	531.71	

It can be seen in Table 1.8 that active power curtailment of PV units with VVC in the co-optimized model is less than the default model under different number of nodes with PV units with VVC.

The amount of cost saving by considering co-optimized model versus default model is also shown in Table 1.8. Since the cost of purchasing active power from PV units is cheaper than substation, the operation cost reduces with less PV curtailment. It is shown in Table 1.8 that the amount of cost saving in all cases are more than 5% cost saving. It is also shown in Table 1.9 that the total amount of absorbed reactive power and Qmax setpoint of Q-V curve of all PV units with VVC is larger in the co-optimized model compared to the default model, while co-optimized model also curtails less active power of PV units with VVC.

Moreover, the dynamic analyses are conducted via a DLL in OpenDSS. The Q-V characteristic of PV units with VVC is programmed and developed in a DLL file and called by OpenDSS. The details of these dynamic models for PV units with VVC are shown in [50]. In all of the case studies, the system is stable with given co-optimized setpoints obtained using IV-based ACOPF. The dynamic response of one of PV units with VVC in the case of 85 nodes with PV unit and VVC is shown in Figure 2.4. At time 0.02 seconds VVC of PV units is activated with given co-optimized setpoints.



Figure 2.4 (a) Voltage magnitude, (b) active power, and (c) reactive power output of one of PV units with VVC

#### 2.3.5 Result: Q-V curve modeling of PV units with VVC in SOCP-based ACOPF:

Both default setpoints and co-optimized setpoints for the SOCP-based ACOPF models are tested on the primary distribution system of the utility partner feeder data. The results are shown in Table 1.10 - Table 1.12. Table 1.10 and Table 1.11 demonstrate the exactness of the obtained solution via the sufficiently small relaxation and linearization error of the proposed SOCP-based ACOPF model. Table 1.12 shows the comparison between SOCP-based ACOPF with default and cooptimized setpoints.

Default setpoints							
NO. of VVC	Max absolute U <sub>ix</sub> difference (p.u.)	Max absolute linearization error for $R_{ijxy}$ (p.u.)	Max absolute linearization error for $I_{ijxy}$ (p.u.)	Max SOC error (p.u.)			
7	2.4E-03	5.0E-06	3.4E-06	1.1E-05			
8	9.3E-05	5.1E-08	9.8E-08	1.1E-07			
9	1.0E-04	5.1E-08	9.8E-08	1.1E-07			
10	2.4E-04	5.1E-08	9.8E-08	1.1E-07			
11	3.9E-04	9.4E-08	9.8E-08	2.0E-07			
12	5.8E-04	2.7E-07	2.0E-07	5.8E-07			

Table 2.3 Maximum relaxation and linearization error of the SOCP-based ACOPF with default setpoints modeling of VVC.

Table 2.4 Maximum relaxation and linearization error of the SOCP-based ACOPF with cooptimized setpoints modeling of VVC.

Co-optimized setpoints								
NO. of VVC	Max absolute $U_{ix}$ difference (p.u.)	Max absolute linearization error for <i>R<sub>ijxy</sub></i> (p.u.)	Max absolute linearization error for <i>I<sub>ijxy</sub></i> (p.u.)	Max SOC error (p.u.)	Max absolute linearization error for Q-V characteristic (kvar)			
7	8.8E-04	3.5E-07	2.0E-07	7.6E-07	0.33			
8	4.8E-04	1.8E-07	1.0E-07	3.9E-07	0.36			
9	7.9E-04	6.8E-08	9.6E-08	1.5E-07	0.27			
10	5.3E-04	1.0E-06	9.8E-08	2.3E-06	0.20			
11	1.2E-03	1.9E-07	1.6E-07	4.2E-07	0.16			
12	1.2E-03	2.1E-07	1.7E-07	4.5E-07	0.21			

Table 2.5 Comparison of the SOCP-based ACOPF with default and co-optimized setpoints.

	D	efault setpoin	ts	Co-optimized setpoints			
NO. of VVC	Active power from PV (kW)	% Active power curtailment	Reactive power from PV (kVAr)	Active power from PV (kW)	% Active power curtailment	Reactive power from PV (kVAr)	Cost- saving (%)
7	91.2	47%	-82.9	131.8	23%	-141.8	6.5%
8	113.8	41%	-93.6	158.1	17%	-122.8	6.4%
9	153.9	28%	-107.1	196.5	8%	-138.7	6.9%
10	182.8	23%	-117.9	229.1	3%	-141.2	5.9%
11	217.9	16%	-131.1	259.0	0%	-153.1	5.6%
12	238.5	13%	-139.7	274.1	0%	-153.2	5.2%

The project team has studied the active and reactive power output for each PV unit having VVC. Figure 2.5 and Figure 2.6 show the active and reactive power output comparison of 8 PV units having VVC.



Figure 2.5 Active power comparison of 8 PV units having VVC for default and co-optimized setpoints models.



Figure 2.6 Reactive power comparison of 8 PV units having VVC for default and co-optimized setpoints models.

The project team also conducts dynamic simulation to evaluate the system stability for the cooptimized setpoints. Figure 2.7 and Figure 2.8 present the dynamic simulation results for voltage and reactive power outputs. For this simulation, VVC are activated at 0.2 second and load has changed at 0.5 second. The dynamic simulation shows that the co-optimized setpoints would not cause any stability issue in the system.



Figure 2.7 Voltage result of the dynamic simulation for the co-optimized setpoints.



Figure 2.8 Reactive power result of the dynamic simulation for the co-optimized setpoints

#### 2.4 Conclusion

In this work, two convex ACOPF models are proposed for three-phase unbalanced distribution systems. For the SOCP-based ACOPF model, a two-stage iterative-based algorithm is developed to solve the proposed SOCP-based ACOPF. The Taylor series approximation is employed to create a linear relationship among the auxiliary variables to make the SOCP-based approach suitable for unbalanced distribution systems. For the IV-based ACOPF model, Taylor series expansion approach is used to linearize the nonlinear power balance injection constraints in the proposed IV-based ACOPF model. The Q-V characteristics of PV units with VVC is considered based on the guidance of IEEE Standard 1547-2018. The proposed IV-based and SOCP-based ACOPF is extended to account for the Q-V characteristic of PV units with VVC with default and optimal settings.

The simulation results of the proposed two convex ACOPF model show that the proposed models can accurately capture the characteristics of the three-phase unbalanced network. The Q-V

characteristic of PV units with VVC is well represented in the two ACOPF models. Besides, the results of the MISOCP-based and IV-based ACOPF models show that the system operation can be improved if settings of VVC are co-optimized in the model due to the flexibility of the optimal settings model to adjust the reactive power output of PV units. The dynamic simulation results confirm that the optimal settings are valid and cause no stability concern for the distribution system.

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## Part II

## **Bilevel Optimization Approaches for Locating Sensors in Distribution Systems**

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#### **Bilevel Optimization Approaches for Locating Sensors in Distribution Systems**

Stochastic fluctuations in power injections from distributed energy resources (DERs) combined with load variability can cause constraint violations (e.g., exceeded voltage limits) in electric distribution systems. To monitor the operation of grids, sensors are placed to measure important quantities such as voltages and power injections. This second part of the report studies the sensor placement problem which seeks to identify locations for installing sensors that can capture all possible violations of voltage magnitude limits. In Chapter 2, we formulate a bilevel optimization problem that minimizes the number of sensors and avoids false sensor alarms in the upper level while ensuring detection of any voltage violations in the lower level.

The nonlinearity of the power flow equations and the presence of binary variables make this problem challenging. Accordingly, to provide the foundation for reformulating the bilevel sensor placement problem in Chapter 2, Chapter 1 develops and analyzes so-called *conservative linear approximations* of the power flow equations, i.e., linear approximations that intend to overestimate or underestimate a quantity of interest in order to enable tractable algorithms that avoid constraint violations. Using a sample-based approach, we compute these conservative linearizations by solving a constrained linear regression problem. We analyze and improve the conservative linear approximations via an iterative sampling approach, optimizing over functions of the quantities of interest, and a sample-complexity analysis. Considering the relationships between the voltage magnitudes and the active and reactive power injections, we characterize the performance of the conservative linear approximations for a range of test cases.

In Chapter 2, we replace the nonlinear power flow equations with conservative linear approximations in order to ensure that the resulting sensor locations and thresholds are sufficient to identify any constraint violations. Additionally, we apply various problem reformulations to significantly improve computational tractability while simultaneously ensuring an appropriate placement of sensors. Lastly, we improve the quality of the results via an approximate gradient descent method that adjusts the sensor thresholds. We demonstrate effectiveness of our proposed method for several test cases that consider networks with several switching reconfigurations.

The work in this part of the report is associated with the following publications:

- P. Buason, S. Misra, and D.K. Molzahn, "A Sample-Based Approach for Computing Conservative Linear Power Flow Approximations," Electric Power Systems Research, vol. 212, no. 108579, November 2022. Presented at 22nd Power Systems Computation Conference (PSCC), June 27 - July 1, 2022.
- P. Buason, S. Misra, and D.K. Molzahn, "A Data-Driven Method to Identify Critical Locations for Sensors in Distribution Grids," in preparation.

### 1. A Sample-Based Approach for Computing Conservative Linear Power Flow Approximations

#### 1.1 Introduction

This chapter develops linear approximations of the AC power flow equations that represent the physical laws governing the voltages and power flows in a power network. These equations are thus an essential part of the mathematical formulations for a variety of planning, optimization, and control problems. The implicit nonlinear nature of the power flow equations often results in computational and theoretical challenges when solving these problems. These challenges lead to the widespread use of simplified power flow representations to improve tractability and enable solution times that are within the problem's latency requirements.

The linear approximations developed in this chapter are essential to the reformulations needed to solve the bilevel sensor placement problems that the focus of Task 2 in this project and are presented in Chapter 2. The accuracy of these linear approximations is closely connected to the quality of the sensor locations and alarm thresholds that result from solving the reformulated bilevel optimization problems. This chapter therefore provides the foundation for the bilevel optimization problems considered in Chapter 2 by presenting a significant amount of detail on these linear approximations. We leave the analysis of their application to the bilevel sensor placement problem to Chapter 2.

Linear approximations, due to their simplicity and efficiency, are among the most widely used class of approximations. Examples of linear approximations include the DC power flow for transmission systems [1], the LinDistFlow for distribution systems [2], and approximations based on the first-order Taylor expansion, such as the power transfer distribution factor (PTDF) representation, among others [3]. Employing these power flow linearizations significantly improves computational tractability, but comes at the price of accuracy which can result in unsafe or sub-optimal decision making. The magnitude of error introduced by a linearization is system and problem specific. However, most linearizations tend to be agnostic to these details and are derived based on general principles. For example, the DC approximation assumes that the voltage magnitudes throughout the system are close to their nominal values and that overall resistive losses are negligible, while the PTDF representation is only accurate within a neighborhood of a given operating point.

To address these drawbacks, [4] introduces the notion of an *optimal adaptive* linearization of the power flow equations. These linearizations are *adaptive* because they are tailored to a specific system and operating range, and *optimal* since they are constructed to minimize their deviation from the nonlinear AC equations with respect to a chosen error metric. A worse-case metric representing the maximum absolute error is considered in [4], whereas [5] minimizes the expected squared error with respect to an underlying probability distribution over the operating range. The mechanisms

proposed to construct these linearizations are also able to *quantify* the error of the resulting approximations, thus allowing for more informed decision making when employing these linearizations.

Various recent studies have proposed methods to construct improved linearizations. In [6], a model that considers voltage magnitude and reactive power in addition to the voltage angle-active power relationship in the DC power flow (PF) is proposed. Although this relationship includes a squared term of voltage magnitude, it is seen as an independent variable, which keeps the relationship linear. In [7], an optimization strategy is proposed to choose an appropriate set of independent variables from a list of available independent variables (e.g.,  $V, V^2, V^3$ , etc., where V is the voltage magnitude). Additionally, data-driven linear approximations have been studied in [7,8]. However, use of these linearizations in power flow-based optimization problems can still lead to constraint violations.

In this chapter, we consider the notion of a *conservative linear approximation (CLA)*, which in addition to being optimal and adaptive, has the feature of being *conservative*. We say that an approximation is *conservative* if it preferentially under- or over- estimates the approximating quantity of interest. The motivation behind conservativeness is a natural one: almost all of our target applications seek to ensure that quantities such as voltages magnitudes, line flows, generator set-points, etc., remain within given upper and lower limits. For instance, bilevel optimization problems (see, e.g., [9]) and optimization problems with discrete variables (e.g., AC unit commitment [10]) include inequality constraints on voltage magnitudes, line flows, and generator outputs. However, directly enforcing these limits by modeling the nonlinear power flow equations (1.1) in these problems leads to significant computational challenges. CLAs address the need for constraint satisfaction during the construction of the linear approximations rather than deferring it to the problem-solving stage of the target application, thus achieving tractability while ensuring constraint satisfaction. CLAs, therefore, not only simplify the enforcement of bounds in the target problem but also enable the use of linearizations whose error properties are better suited for bound enforcement.

To construct these linear approximations, we propose a sample-based approach where the CLAs are obtained as the solution to a constrained regression problem. This differs from the iterative optimization-based approach of [4] and the polynomial chaos method used in [5], although it is closely related to the latter. The main reason for this choice is scalability, as sample-based regression problems are efficient to solve and can exploit parallelizability and high-performance distributed computing hardware. Further, we propose a basic sample-selection procedure that significantly boosts convergence speed and a Monte-Carlo validation in each selection step for obtaining theoretically guaranteed confidence bounds.

To further improve the quality of the approximations, we propose a method to optimally modify the quantity being approximated while retaining all of the desirable features of the CLA. For example, instead of approximating the voltage magnitude at a bus as a linear function of the active and reactive power injections, our method will automatically determine a different function (such as a monotonic quadratic function) of voltage magnitude that has a smaller linearization error. The procedure to determine the modified function is directly built into the constrained regression used

to construct the CLA. While imposing a small computational cost, this can significantly reduce the approximation error. We also show that when the modified function is monotonic in the quantity of interest, this modification introduces no additional complexity when the constructed CLA is used in problems relevant to our target applications.

In summary, the main contributions of this chapter are:

(*i*) A CLA formulation that is adaptive to the relevant system and operating range, optimal with respect to an error metric, and conservative in that it under- or over-estimates the quantity of interest.

(*ii*) A procedure to modify the quantity of interest being approximated that improves error without sacrificing beneficial properties of the CLA.

(*iii*) A data-driven approach to construct the CLAs using a constrained regression formulation and a sample selection method that improves the convergence rate.

(*iv*) Numerical analysis of the CLA for a variety of test cases and a demonstration of its application to the optimal power flow problem.

This chapter is organized as follows. Section 1.2 provides background on the power flow equations. Section 1.3 formulates and solves an optimization problem for computing CLAs using a sample-based approach. Section 1.4 presents our numerical tests. Section 1.5 concludes the chapter and discusses future work.

#### **1.2 The Power Flow Equations**

Consider an *n*-bus power system with a fixed network topology. The sets of buses and lines are  $\mathcal{N} = \{1, ..., n\}$  and  $\mathcal{L}$ , respectively. One bus in the system is specified as the slack bus (also known as the reference bus), where the voltage magnitude and angle is set to be  $1 \angle 0^\circ$  per unit. Each of the remaining buses is classified as either a PV bus, which specifies the active power and the voltage magnitude, or a PQ bus, which specifies the active and reactive power. The set of buses that are neighbors to bus *i* is denoted as  $\mathcal{N}_i := \{k \mid (i,k) \in \mathcal{L}\}$ . Subscript  $(\cdot)_i$  denotes that quantity at bus *i*, and subscript  $(\cdot)_{ik}$  denotes that quantity from or connecting bus *i* to *k*. The power flow equations at bus *i* are:

$$P_i = V_i^2 G_{ii} + \sum_{k \in \mathcal{N}_i} V_i V_k (G_{ik} \cos \theta_{ik} + B_{ik} \sin \theta_{ik}), \qquad (1.1a)$$

$$Q_i = -V_i^2 B_{ii} + \sum_{k \in \mathcal{N}_i} V_i V_k (G_{ik} \sin \theta_{ik} - B_{ik} \cos \theta_{ik}), \qquad (1.1b)$$

where  $V_i$  is the voltage magnitude at bus i,  $P_i(Q_i)$  is the active (reactive) power injection at bus i,  $\theta_{ik} := \theta_i - \theta_k$  is the voltage angle difference between buses i and k, and  $G_{ik}(B_{ik})$  is the real (imaginary) part of the network admittance matrix entry associated with buses i and k. Equations (1.1a)–(1.1b) are called the full AC-PF equations.

#### **1.3** Problem Formulation and Solution Approach

This section introduces the conservative linear approximations (CLAs), formulates a constrained regression problem to compute the CLAs using a sample-based approach, and presents methods for improving the accuracy of the CLAs by modifying a function of the quantity being approximated and by using a sample selection procedure to speed up calculations.

Without loss of generality and for ease of exposition, we use the voltage magnitude at a bus as our quantity of interest and consider the CLAs that express the voltage magnitude as a function of active and reactive power injections.

#### 1.3.1 Conservative linear approximations of the power flow equations

Linear approximations of the power flow equations are widely used to reduce the complexity of formulating many operation and planning problems. We propose to develop linear approximations that are *conservative*, meaning that the computed linear approximation is greater or equal to the actual quantity for an overestimating approximation or the other way around for an underestimating approximation.

Using voltage magnitude as our quantity to approximate and active and reactive power injections as input variables, our goal is to compute approximations that satisfy:

$$V_i - g_i(\mathbf{P}, \mathbf{Q}) \le 0 \ (\ge 0);$$
 if overestimate (underestimate) (1.2)

where

$$g_i(\mathbf{P}, \mathbf{Q}) = a_{i,0} + \mathbf{a}_{i,1}^T \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}$$
(1.3)

is a linear function of active power injections at PQ buses ( $\mathbf{P}$ ) and reactive power injections at PV and PQ buses ( $\mathbf{Q}$ ).

#### **1.3.2** Data-driven power flow approximations

Instead of an analytic or purely optimization-based approach, we propose a data-driven approach to compute the CLAs. The data-driven approach is highly scalable, can utilize advanced computing hardware, permits a parallel implementation, and can take advantage of improved sensing and communication technology for on-the-fly updates.

Using our example, we compute the data-driven power flow approximations by randomly sampling load demands (or generator outputs) within a given operating range and calculating the corresponding bus voltages by solving the power flow equations for each of these samples. The samples for load demands are obtained using a given probability distribution  $\mathbb{P}_{\mathscr{S}}$  over a specified operating range  $\mathscr{S}$ . For example, one may consider  $\mathscr{S} = \{P_{L_d}^{\min} \leq P_{L_d} \leq P_{L_d}^{\max}, Q_{L_d}^{\min} \leq Q_{L_d} \leq Q_{L_d}^{\max} \text{ for all } d \in \mathscr{N}_{\mathscr{D}}\}$  where  $(\cdot)_{L_d}$  denotes the load demand,  $\mathscr{N}_{\mathscr{D}}$  is the set of load buses,  $\mathbb{P}_{\mathscr{S}}$  is

the uniform distribution, and superscript max (min) is an upper (lower) limit. The power injections at PQ buses change with varying load demands change, which also affects the reactive power injections at PV buses (see (1.3)).

In this chapter, we approximate voltage magnitudes as linear functions of the power injections. Using uniformly generated samples, we propose to minimize errors (e.g., the 1-norm error) between the CLA and the samples. Using *M* power flow samples, we consider the following regression problem that computes a CLA while minimizing a chosen loss function  $\mathcal{L}(\cdot)$  representing the approximation error.

$$\min_{g(\cdot)} \sum_{m=1}^{M} \mathscr{L}(V_{i,m} - g_i(\mathbf{P}_m, \mathbf{Q}_m))$$
(1.4a)

s.t. 
$$V_{i,m} - g_i(\mathbf{P}_m, \mathbf{Q}_m) \le 0$$
; if overestimate, (1.4b)

$$V_{i,m} - g_i(\mathbf{P}_m, \mathbf{Q}_m) \ge 0$$
; if underestimate, (1.4c)

for all m = 1, 2, ..., M. Subscript *m* denotes the *m*-th sample. All bold quantities are vectors. The set of equations (1.4b) (or (1.4c) as chosen) enforces conservativeness and ensures that the CLA remains above (or below, as appropriate) the actual voltage magnitudes for all samples. Examples of  $\mathscr{L}(\cdot)$  include the  $\ell_1$  and  $\ell_2$  loss functions.

Like any linear function, we note that the computed CLAs are defined for all inputs, i.e., every choice of power injections  $\mathbf{P}_m$ ,  $\mathbf{Q}_m$  is mapped to a voltage magnitude  $g_i(\mathbf{P}_m, \mathbf{Q}_m)$ . Conversely, the nonlinear power flow equations can be insolvable for some power injections, i.e., there may be power injections without corresponding voltages. When constructing the CLAs, we ignore any sampled power injections for which the power flow equations are insolvable. Thus, our goal is to compute the CLAs with respect to power injections for which the power flow equations are solvable.

#### **1.3.3** Optimal output function design

For many applications, one can significantly reduce the linearization error for some quantity of interest by instead approximating an appropriately chosen function of that quantity. Using our voltage magnitude example, instead of approximating the voltage magnitude  $V_i$  itself, we propose to linearly approximate a function of the voltage magnitude,  $f(V_i)$ , in terms of **P** and **Q**. We call the function  $f(\cdot)$  an *output* function. To motivate the idea of an output function, we consider a simple two-bus system and examine the relationship between the voltage magnitudes and the power injections.

Multiplying (1.1a) by  $B_{ik}$  and (1.1b) by  $G_{ik}$  to eliminate  $\cos \theta_{ik}$ , the power flow equations for a two-bus system are:

$$P_1B_{12} + Q_1G_{12} = V_1^2(G_{11}B_{12} - G_{12}B_{11}) + V_1V_2(B_{12}^2 + G_{12}^2)\sin\theta_{12}, \qquad (1.5a)$$

$$P_2B_{21} + Q_2G_{21} = V_2^2(G_{22}B_{21} - G_{21}B_{22}) + V_1V_2(B_{21}^2 + G_{21}^2)\sin\theta_{21}.$$
 (1.5b)

Consider the case where the admittance matrix is symmetric, bus 1 is a slack bus, and bus 2 is a PQ bus. Adding (1.5a) to (1.5b) gives:

$$V_2^2(G_{22}B_{21} - G_{21}B_{22}) = (P_1 + P_2)B_{12} + (Q_1 + Q_2)G_{12} + \beta(G_{11}B_{12} - G_{12}B_{11}), \quad (1.6)$$

where  $\beta = -V_1^2$  is a constant, which is typically -1. Here, we model the power injections at both buses 1 and 2 ( $P_1$ ,  $P_2$ ,  $Q_1$ , and  $Q_2$ ) as input variables for the linear approximation (1.3). Thus, (1.6) shows that the relationship between  $V_2$  and power injections at buses 1 and 2 is quadratic. Hence, a linear output function  $f(V_2) = V_2$  will result in a non-zero approximation error, whereas using a quadratic output function  $f(V_2) = V_2^2$  could result in zero approximation error. The relationship in (1.6) is specific to the two-bus system, and more complicated systems will have non-zero approximation errors even with quadratic output functions. However, the two-bus example motivates the use of non-linear output functions. While there are many possible output functions, this chapter considers two natural choices: (i) polynomials and (ii) piecewise linear functions. Polynomial output functions are:

$$f(V_i) = b_n V_i^n + b_{n-1} V_i^{n-1} + \dots + b_1 V_i,$$
(1.7)

where  $b_k$  is a polynomial coefficient for all k = 1, 2, ..., n. Given *n* segments, piecewise linear output functions are:

$$f(V_i) = c_{0j} + c_{1j}V_i, \quad \text{if } p_j \le V_i \le p_{j+1}, \ 0 \le j \le n-1, \tag{1.8}$$

where  $p_i$  are the specified breakpoints, and  $c_{0i}$  and  $c_{1i}$  are coefficients corresponding to each piece.

To avoid calculating non-meaningful and trivial solutions, enable efficient computation of the CLA using the constrained regression in (1.4), and preserve the advantages of using the CLA in our target applications, we enforce the following characteristics on the output function  $f(\cdot)$ :

**Characteristic 1** *The function*  $f(\cdot)$  *is monotonically increasing within a specified range.* 

**Characteristic 2** The range of  $f(\cdot)$  is bounded below; i.e.,  $f(\overline{V_i}) - f(\underline{V_i}) \ge c$ , for some c > 0, where  $\overline{(\cdot)_i}$  and  $\underline{(\cdot)_i}$  denote the maximum and minimum values of that quantity at bus *i*, respectively, among the sampled power flow solutions.

Characteristic 1 means that  $f(\cdot)$  is invertible. More importantly, it is required to ensure that when the CLA is used in an optimization problem, the resulting constraint *remains linear* even when the output function  $f(\cdot)$  is nonlinear. This idea is demonstrated in detail using the optimal power flow (OPF) problem as an example application in Section 1.3.6. Characteristic 2 is necessary to avoid trivial output functions since  $f(\cdot) = 0$  will always result in zero approximation error.

For piecewise linear  $f(\cdot)$ , we impose additional characteristics to make the approximation well behaved.

**Characteristic 3** A piecewise  $f(\cdot)$  is continuous within an operating range.

**Characteristic 4** Let p be the set of all breakpoints. The slope of a piecewise function is bounded from above except at breakpoints; i.e.,  $f'(V) \le d$ ,  $\forall V \notin p$ , for some specified  $d \ge 0$ , where f'(V) denotes the first derivative of f(V).

Characteristic 3 ensures continuity of the piecewise function. Characteristic 4 prevents the piecewise function from having sharp changes within its range, especially towards the end of its range. This is also a desirable property for the polynomial output functions. The number and locations of the breakpoints in the piecewise linear function also impact the accuracy of the corresponding CLA. We chose the breakpoints such that each segment of the piecewise linear function contains the same number of sampled voltages.

It is also possible to include the following additional characteristic which may be beneficial for certain applications.

#### Characteristic 5 (Optional) An output function is either convex or concave.

We note that the output function of each voltage magnitude can be different. Let  $f_i(V_i)$  denote the output function of  $V_i$ . We rewrite the regression in (1.4) incorporating the above characteristics (when applicable) to obtain the full version of the constrained regression to construct a CLA at bus *i*:

$$\min_{f_i(\cdot),g_i(\cdot)} \sum_{m=1}^M \mathscr{L}\left(f_i(V_{i,m}) - g_i(\mathbf{P}_m, \mathbf{Q}_m)\right)$$
(1.9a)

s.t. 
$$f'_i(V_{i,m}) \ge 0,$$
 (1.9b)

$$f_i(\overline{V_i}) - f_i(\underline{V_i}) \ge c,$$
 (1.9c)

$$f'_i(V_i) \le d$$
 (if piecewise), (1.9d)

$$f_i(V_i)$$
 is continuous (if piecewise), (1.9e)

$$f_i(V_{i,m}) - g_i(\mathbf{P}_m, \mathbf{Q}_m) \le 0; \text{ (if overestimate)}, \tag{1.9f}$$

$$f_i(V_{i,m}) - g_i(\mathbf{P}_m, \mathbf{Q}_m) \ge 0;$$
 (if underestimate), (1.9g)

for all m = 1, ..., M. Note that the decision variables in (1.9) are the coefficients of  $f_i(\cdot)$  and  $g_i(\cdot)$ . We observe that when the output function  $f_i(\cdot)$  can be linearly parameterized, which is the case for both polynomial and piecewise linear output functions, the constraints (1.9b)–(1.9g) are *linear* in these parameters (e.g., coefficients of the polynomial). When the loss function is chosen to be  $\mathscr{L}(x) = |x|$  representing an  $\ell_1$ -loss, the regression (1.9) can be cast as a *linear program* by using the standard slack formulation for minimizing the  $\ell_1$ -norm, which enables an efficient solution. For the rest of this section and for our experiments, we will select  $\mathscr{L}(x) = |x|$ .

We define the minimum value of the objective function in (1.9a) as the **CLA error**. We also define the **actual error** as the error obtained by inverting the output function. Since characteristic 1 allows us to invert  $f_i(\cdot)$ , we can compute the voltage predicted by the CLA for each sample,  $V_{i,m}^C$ , as

$$V_{i,m}^{C} = f_{i}^{-1} \left( g_{i}(\mathbf{P}_{m}, \mathbf{Q}_{m}) \right).$$
(1.10)



Figure 1.1: An example of the sample selection method. The red circles and the blue stars are the previous and the new samples, respectively. The top (bottom) lines illustrate over- (under-) estimating CLAs. The right picture shows that only new samples which violate the previous CLA are included to compute the updated CLA. Specifically, the stars with the dashed circles (squares) are included for the updated over- (under-) estimating CLA.

The average actual error,  $e_{a,i}$ , is defined as

$$e_{a,i} = \frac{1}{M} \sum_{m=1}^{M} |V_{i,m}^C - V_{i,m}|.$$
(1.11)

As the name suggests, we would ideally want to directly minimize  $e_{a,i}$  in (1.11) as opposed to the objective function (1.9a). However, this can make (1.9) a (possibly non-convex) non-linear program that is much more challenging to solve than the linear program in (1.9). We therefore instead use the CLA error (1.9a) as a proxy for minimizing the actual error (1.11) that is significantly faster to compute. Our numerical experiments support the effectiveness of this approach.

#### 1.3.4 Sample selection method and confidence bound

Since the calculated CLA is based on random samples, it is possible that newly drawn samples could be on the wrong side of CLA (e.g., a new sample may have a higher voltage than the overestimate provided by a CLA). Increasing the number of samples M used to compute the CLA in (1.9) improves confidence with respect to its conservativeness.

To reduce the number of samples needed to achieve a certain amount of confidence, we propose a sample selection method that iteratively updates a CLA using only violating samples (see Fig. 1.1). To perform the sample selection, we iterate between:

- 1. Drawing additional samples and adding points using the sample selection method, which only considers points that violate the CLA calculated in the previous iteration.
- 2. Solve (1.9) to update the CLA. Repeat these steps until satisfying termination conditions.

All sampled points in step 1) can be further analyzed, so only meaningful points are added in each iteration. Note that the overestimating CLA and the underestimating CLA are updated separately based on violated points, i.e., a point that violates the overestimating CLA does not violate the underestimating CLA. Thus, that point is not included in the re-calculation of the underestimating CLA. At each step of the iteration, we use  $M^{out}$  samples to perform an out-of-sample validation which allows us to provide confidence bounds on the over- or under- estimation property of the CLA. Let  $e^{out}$  be the fraction of the  $M^{out}$  samples for which there is a violation of the over- or under- estimation property of the CLA in the out-of-sample testing. For a randomly drawn set of (**P**,**Q**), let  $V_i$  denote the actual voltage and let  $V_i^C$  denote the predicted voltage using the CLA. Then the probability  $\mu^{err} = \mathbb{P}_{\mathscr{S}}(V_i^C < V_i)$  that a randomly drawn sample violates the over- or under- estimation property for the CLA can be bounded using the confidence bound

$$\mathbb{P}\left(\mu^{err} > e^{out} + \delta\right) \le e^{-2\delta^2 M^{out}},\tag{1.12}$$

which can be derived using the Hoeffding inequality [11].

#### **1.3.5** Computation time

Using the CLA instead of the non-linear AC-PF equations can significantly improve the computational efficiency of problems for many applications. The regression computing the CLA (1.9) can be pre-computed offline given the system topology, line limits, range of load demands, and other predefined constraints. Moreover, this offline computation can be parallelized for each quantity of interest independently. Since the CLAs are constructed separately for each constrained quantity, CLA-based constraints can be imposed adaptively using a scenario and constraint generation approach that only enforces a constraint after determining that it may be binding. Conversely, when using the implicit AC-PF equations, if an operating scenario has even a single binding constraint, then all AC-PF equality constraints must be added for that scenario.

#### **1.3.6** Application: Optimal power flow

While the advantages of using the CLA are best seen in more complex optimization problems such as bilevel problems [9] and the AC unit commitment problem [10], a detailed analysis of such applications is out of scope of this chapter. Instead, we use a simplified AC-OPF problem as a proof-of-concept demonstration of CLAs. The simplified OPF problem we consider here is identical to the classical OPF problem except that the line flow limits are neglected such that enforcing bus voltage limits is the primary goal in order to match the exhibition earlier in this chapter. We emphasize that although we do not focus on line flow limits here, one may include them by constructing CLAs that approximate the line flows.

Let  $\mathcal{N}_S$ ,  $\mathcal{N}_{PV}$ , and  $\mathcal{N}_{PQ}$  denote the sets of slack, PV, and PQ buses, respectively. To compute the CLAs for this simplified OPF problem, we change the inputs of the linear function in (1.3) to be  $P_i$  and  $V_i$  at all  $i \in \mathcal{N}_{PV}$ . The approximated quantities are (i)  $Q_i$ , for all  $i \in \mathcal{N}_{PV} \cup \mathcal{N}_S$ , (ii)  $V_i$  for all  $i \in \mathcal{N}_{PQ}$ , and (iii)  $P_s$  at a slack bus  $s \in \mathcal{N}_S$ . Let  $\tilde{g}(\mathbf{P}, \mathbf{V})$  ( $g(\mathbf{P}, \mathbf{V})$ ) be the function that overestimates (underestimates) quantities of interest,  $\tilde{f}(\cdot)$  ( $f(\cdot)$ ) be the polynomial output function corresponding to  $\tilde{g}(\mathbf{P}, \mathbf{V})$  ( $g(\mathbf{P}, \mathbf{V})$ ), and  $\mathcal{N}_{\mathcal{G}}$  be the set of all generators. We denote the power generation and generation cost as  $P_G$  and c, respectively. By formulating voltage and power generation limits with CLAs, the simplified OPF problem becomes:

$$\min \sum_{k \in \mathcal{N}_{\mathcal{G}}} c_k P_{G_k} \tag{1.13a}$$

s.t. 
$$\widetilde{f}_i(\overline{V}_i) \ge \widetilde{g}_i(\mathbf{P}, \mathbf{V}), \quad \underline{f}_i(\underline{V}_i) \le \underline{g}_i(\mathbf{P}, \mathbf{V}), \quad (1.13b)$$

$$\widetilde{f}_{j}(\overline{Q}_{j}) \ge \widetilde{g}_{j}(\mathbf{P}, \mathbf{V}), \quad \underline{f}_{j}(\underline{Q}_{j}) \le \underline{g}_{j}(\mathbf{P}, \mathbf{V}),$$
(1.13c)

$$\widetilde{f}_{s}(\overline{P}_{s}) \geq \widetilde{g}_{s}(\mathbf{P}, \mathbf{V}), \quad \underline{f}_{s}(\underline{P}_{s}) \leq \underline{g}_{s}(\mathbf{P}, \mathbf{V}), \quad (1.13d)$$

$$P_s = \widetilde{g}_{s1}(\mathbf{P}, \mathbf{V}), \tag{1.13e}$$

$$P_{G_k}^{\min} \le P_{G_k} \le P_{G_k}^{\max}, \ Q_{G_k}^{\min} \le Q_{G_k} \le Q_{G_k}^{\max},$$
(1.13f)

for all  $i \in \mathcal{N}_{PQ}$ ,  $j \in \mathcal{N}_{PV} \cup \mathcal{N}_S$ ,  $s \in \mathcal{N}_S$ , and  $k \in \mathcal{N}_G$ . We call (1.13) a **CLA-OPF** problem. Constraints (1.13b)–(1.13d) ensure that all approximated quantities are within their limits. The overestimated value of the power generation at the slack bus in (1.13e) is used in the objective function. The resulting CLA-OPF is a linear program that nevertheless ensures satisfaction of the inequality constraints in the non-linear AC-OPF problem so long as the computed conservative linear approximations are indeed conservative.

#### 1.4 Simulation and Results

In this section, we use numerical experiments on a number of test cases to analyze various properties of the CLA, demonstrate the advantages of output function design and sample selection procedures, and demonstrate the effectiveness of the CLA in enforcing constraints using the simplified OPF.

The test cases we use in the simulations are a 6-bus system (adapted from *case6ww*), whose parameters can be found in [12], the 33-bus system *case33bw* from MATPOWER [13], and the IEEE 300-bus system. Note that the samples used to create each table and figure below are drawn independently, and 5000 samples are drawn in each test unless stated. All load demands vary within -50% to 200% of the values given in the MATPOWER test systems, except for the OPF simulations where the loads are fixed and the generator outputs vary. Only samples for which the power flow solver converges are included in CLA calculation, and the voltage magnitude at one representative bus per system is chosen for illustration. The loss function represents the  $\ell_1$  error.

#### 1.4.1 Range constraint for polynomial output functions

As discussed in Section 1.3.3, a lower bound c is enforced on the range of the output function (characteristic 2) to avoid trivial solutions. The value of c is an arbitrary parameter chosen when constructing a CLA, and it is therefore important to analyze the impact that this choice has on the CLA's linearization error.

For this purpose, Fig. 1.2(a) shows a representative example of the relationship between c and



Figure 1.2: Plot of (a) CLA errors, (b) actual errors at bus 6 in the 6-bus system using the quadratic output functions. The solid and dashed lines represent the errors from underestimating and overestimating the voltage magnitude  $V_6$ , respectively

the CLA error for the 6-bus system. We observe that the CLA errors vary linearly with the value of c. This is reasonable since decreasing c relaxes constraint (1.9c), leading to reductions in the optimal objective value of (1.9). Conversely, Fig. 1.2(b) shows that the actual errors obtained using (1.10) do *not* depend on the value of c as the plots in this figure are constant. The results from cubic and quartic output functions, which are not shown in this chapter, are similar. This analysis suggest that the value of c can be arbitrarily chosen and does not affect the optimal solution for over/underestimated voltage magnitude.

#### 1.4.2 Error reduction with polynomial output functions

We next investigate the effect of using polynomials (1.7) with various degrees *n* as output functions as discussed in Section 1.3.3. While the CLA error is guaranteed to be non-increasing with the degree of the polynomial, the same is not necessarily true for the actual error in (1.11).

Moreover, as discussed above, the choice of c does not affect the actual error of the CLAs. We arbitrarily set c to 1.5. The results from the first-order Taylor approximation around the operating point (given in the MATPOWER test cases) are also included for reference. The results are reported in Table 1.1.

Table 1.1 presents the averages of the CLA errors and the actual errors from the quartic, cubic, quadratic, and linear functions for the CLA associated with the voltage magnitude at bus 6 in the 6-bus system and bus 33 in the 33-bus system. Note that using a linear output function is equivalent to not designing an output function. The average CLA errors are smaller when the degree of the polynomial is higher. This is due to the fact the optimal solution of a lower-degree polynomial is a feasible solution for the higher-degree polynomial. The percentage reductions in the average actual

e		Average errors ( $ imes 10^{-2}$ ) in per unit				
Cas	Function	Un	Iderestimate	Overestimate		
-		CLA	CLA Actual (*)		Actual (*)	
e-bus	Quartic	4.57	1.451 (54.70%)	2.29	0.655 (47.93%)	
	Cubic	4.63	1.358 (57.60%)	2.29	0.657 (47.77%)	
	Quadratic	4.64	1.327 (58.57%)	2.33	0.662 (47.38%)	
	Linear	11.73	3.203 (0%)	4.61	1.258 (0%)	
	Taylor		1.8	00 <sup>§</sup>		
	Quartic	0.90	0.059 (58.74%)	0.32	0.021 (51.16%)	
$p_{W}$	Cubic	0.91	0.059 (58.74%)	0.32	0.021 (51.16%)	
case33	Quadratic	0.91	0.059 (58.74%)	0.32	0.021 (51.16%)	
	Linear	2.15	0.143 (0%)	0.64	0.043 (0%)	
	Taylor		0.2	59 <sup>§</sup>		

Table 1.1: Results comparing average CLA and actual errors from CLAs at bus 6 in the 6-bus system and at bus 33 in the 33-bus system

\*The values in () show the percentage reductions in the actual errors compared to the actual errors from linear function.

<sup>§</sup> The errors from first-order Taylor approximation are the average absolute value of the errors caused by the approximation.

errors with respect to the linear function are also shown in Table 1.1. The average actual errors from the quadratic, cubic, and quartic functions are significantly reduced from the linear function (from 47% to 59% in both systems) with the quadratic and higher-degree polynomials showing similar performance. In comparison, the first-order Taylor approximation is not conservative and has significantly larger error.

We further analyze the behavior of all tested functions by depicting them in Fig. 1.3. To enable a consistent comparison, each function is adjusted using the offset  $a_0$ , which is the constant term for  $g(\mathbf{P}, \mathbf{Q})$  defined in (1.3). The functions corresponding to overestimates of the voltage magnitudes are shown for a range from 20% below to 20% above the extreme sampled values. Since both CLA and actual errors for quadratic, cubic, and quartic functions are fairly close (refer to Table 1.1), these functions are almost identical within the sampled operating range but are all quite different from the linear function. However, they are different functions as clearly seen outside the sampled operating range. For this illustration, we intentionally select a large range of voltage magnitudes to more clearly show the curvature of each function, but note that similar behavior occurs for smaller voltage magnitude ranges.

To further illustrate how the output function f(V) transforms the samples, we compute over/underestimates of the voltage magnitudes for each sample. Fig. 1.4 shows the results for the linear, quadratic, cubic, and first-order Taylor approximation functions. The horizontal axis shows the voltage magnitude from the power flow solution while the vertical axis shows the predicted



Figure 1.3: Results from the overestimating CLAs for the voltage magnitude at bus 6,  $V_6$ , in the 6-bus system for polynomials of different degrees. The black vertical lines show the minimum and maximum observed voltage magnitudes.

voltage magnitude from the CLA. The quadratic and cubic output functions make the behavior of the over/underestimated voltages closer to linear, resulting in reduced approximation errors. The figure also shows the conservative property of the CLAs when compared to the first-order Taylor approximation.

#### 1.4.3 Error reduction with piecewise linear output functions

Section 1.3.3 also discusses piecewise linear output functions. The accuracy of a piecewise linear function depends on the number of breakpoints. In this simulation, the number of breakpoints is 100, the number of samples is 5000, and each piece contains the same number of samples. The value of d from Characteristic 4 is set to 50.

Fig. 1.5 compares a piecewise linear function with quadratic, cubic, and quartic functions. The results from the piecewise linear function are similar to the results from other functions, especially at the middle of the range of voltage magnitudes where the power flow solutions from samples are most dense (refer to Fig. 1.4). On the other hand, when the voltage magnitude is near the minimum or maximum value, the piecewise linear function has a larger difference with respect to the other functions. This is due to an overfitting phenomenon resulting from the larger number of free parameters in the piecewise approximation compared to the polynomials. The limit imposed on the maximum derivative in (1.9d) serves as a regularizing constraint for the piecewise linear case.

Table 1.2 shows the average CLA errors and the average actual errors when varying the number of breakpoints. Results using one breakpoint are the same as those from a linear function. Increasing the number of breakpoints continually reduces the CLA errors, but the actual errors saturate quickly.



Figure 1.4: Plots of results from (a) linear, (b) quadratic, (c) cubic, and (d) first-order Taylor approximation functions at bus 6 in the 6-bus system. The blue and red points represent underestimates and overestimates of the voltage magnitude, respectively, in per unit. The first-order Taylor approximation neither overestimates nor underestimates the voltage magnitude. The black line at  $45^{\circ}$  represents a perfect fit.

#### **1.4.4** Sample selection

The accuracy of the CLA depends on the samples input to (1.9). Increasing the number of samples improves the accuracy of the CLA but reduces the computational tractability of (1.9), thus motivating the sample selection method discussed in Section 1.3.4 which iteratively updates the CLA with violated samples. To demonstrate the effectiveness of this method, Fig. 1.6 presents the reduction in number of violated samples using an *out-of-sample* validation after applying the sample selection method for the 6- and 33-bus systems. Drawing 10000 samples for each iteration, the sample selection method reduces the percentage of violated samples from 0.49% to 0.04% after 10 iterations for


Figure 1.5: The results from overestimates of the voltage magnitude at bus 6 in the 6-bus system with different polynomials and a piecewise linear function. The black vertical lines show the minimum and maximum observed voltage magnitudes obtained from samples. The horizontal axis corresponds to the per unit voltage magnitude and the vertical axis is the output function. The results from underestimates of the voltage magnitude are similar.

Table 1.2:	Results	comparing	errors	from	piecewise	linear	approximation	with	different	number	of l	oreak-
points at bu	us 6 in th	ne 6-bus sys	stem an	d bus	33 in the	33-bus	system					

e	Number	Average errors ( $\times 10^{-2}$ ) in per unit					
Cas	of	Under	estimate	Overestimate			
•	Breakpoints	CLA	Actual	CLA	Actual		
	1	12.59	3.331	4.56	1.215		
snq	10	4.80	1.335	2.59	0.723		
-9	100	4.11	1.251	2.27	0.674		
	500	4.05	1.251	2.20	0.682		
M	1	1.99	0.137	0.62	0.043		
33b	10	0.83	0.057	0.32	0.022		
ase.	100	0.76	0.053	0.30	0.020		
J	500	0.72	0.051	0.28	0.019		

the 6-bus system and from 2.40% to 0.21% after 10 iterations for the 33-bus system. The results show that sample selection significantly improves the convergence rate of the error compared to random sampling.

#### 1.4.5 Scalability: CLA in larger systems

We have so far shown results from the 6- and 33-bus systems, which represent a small transmission system and a small radial distribution system, respectively. We next use the larger IEEE 300-bus system provided in MATPOWER to show the consistency of some of the properties analyzed in the



Figure 1.6: Plots showing the percentage of violated samples (vertical axis) after applying sample selection in each iteration (horizontal axis). The red lines and the blue lines show the results with and without applying sample selection, respectively, considering the same number of additional samples in each case. Note that each iteration uses different sets of samples.

previous sections. This simulation focuses on bus 250. Only average actual errors are computed as they are the quantities of interest. The piecewise linear function has 1000 breakpoints.

Table 1.3: Results comparing actual errors from different CLA functions at bus 250 in the IEEE 300-bus system

	Average actual errors ( $\times 10^{-3}$ ) in per unit					
Function	Underestimate (*)	<b>Overestimate</b> (*)				
Quartic	5.961 (27.79%)	4.884 (17.65%)				
Cubic	5.957 (27.84%)	4.676 (21.16%)				
Quadratic	5.801 (29.73%)	4.650 (21.60%)				
Piecewise	6.026 (27.00%)	4.768 (19.61%)				
Linear	8.255 (0%)	5.931 (0%)				
Taylor	2	2.8 <sup>§</sup>				

\*The values in () show the percentage reductions in the actual errors compared to the actual errors from a linear function.

<sup>§</sup> The errors from the first-order Taylor approximation are the average absolute value of the errors caused by the approximation.

Table 1.3 compares the average actual errors for different CLA functions. For the voltage underestimation, the piecewise linear functions and other polynomials with degree two or more reduce the average actual errors by 27% to 29.73% from the linear function. Similarly, these errors reduce by 17.65% to 21.60% for the voltage overestimation. These output functions thus notably increase the accuracy in the voltage approximation. For the first-order Taylor approximation, which is not



Figure 1.7: Sample selection method for CLA at bus 250 in the 300-bus system.

conservative, the average errors are significantly greater, approximately by a factor of three to four compared to the other CLA functions.

Fig. 1.7 shows the results from the sample selection method. We use sufficient iterations to ensure that the probability of violation is less than 1%. In each iteration, 10000 samples are randomly drawn for out-of-sample testing. The percentage of violated points reduces from 7.39% to 0.92% after 12 iterations of sample selection. Although the larger system requires more samples to achieve a low violation probability, the iterative sample selection approach still reduces the violation probability significantly faster than using random samples.

#### 1.4.6 Simplified OPF

The OPF problem serves as a simple example for an application of the CLA method. We solve the simplified OPF problem (discussed in Section 1.3.6) with different approximation techniques (e.g., DC-OPF and CLA-OPF) to compare with the solution to the AC-OPF problem. We also compare our results with linear approximation OPF (LA-OPF), which minimizes the 1-norm error and follows characteristics 1 and 2, but is not necessarily conservative. For the LA-OPF and the CLA-OPF, we choose quadratic output functions.

Table 1.4 compares different approximations of the nonlinear AC-OPF for the 6-, 9-, and 14-bus systems. The first row in each cell of this table reports the actual cost corresponding to the AC-PF feasible solution obtained by using the control set points prescribed by the various OPF formulations. The second row in each cell gives the maximum voltage magnitude violation for the solution. We see that only the CLA-based control actions do not have violations, which is attributed to the conservative nature of the CLAs. With the approximated voltage magnitude of 1 per unit (pu), the DC-OPF gives the maximum voltage violation of 0.029 pu in *case6ww*. For the LA-OPF, the maximum voltage violation is 0.004 pu in *case14*. By minimizing an upper bound on the cost, CLA-OPF is also able to obtain optimal or near-optimal solutions with lower costs than the other approxima-

		Case	
	саѕебww	case9	case14
AC-OPF	2986.04	1456.83	5368.30
DC-OPF	2995.15 (0.31%)	1502.82 (3.16%)	5368.52 (0.004%)
Violation	V (0.029 pu)	-	-
LA-OPF	2987.28 (0.04%)	1468.48 (0.88%)	5368.52 (0.004%)
Violation	-	-	V (0.004 pu)
CLA-OPF	2987.51 (0.05%)	1469.68 (0.88%)	5368.52 (0.004%)
Violation	-	-	-

Table 1.4: Results comparing solutions from AC-, DC-, LA-, and CLA-OPF

tions. The optimality gaps from the CLA-OPF are comparable to those from the LA-OPF, and they are significantly smaller than the DC-OPF in *case6ww* and *case9*.

#### 1.5 Conclusion

This chapter proposes a power flow linearization approach that intends to be conservative (overestimate/underestimate a quantity of interest) and adaptive (due to a sample-based approach, it is tailored to a specific system and an operational range). Our proposed method is not limited to a linear output function as we introduce a higher-degree polynomial as an independent variable, which maintains linearity. The additional degrees of freedom from a higher-degree polynomial enable us to exploit a closer-to-linear relationship to other known quantities (e.g., power injections). Our numerical results demonstrate improvements in accuracy relative to other linear approximations while still being conservative. Additionally, the results in Section 1.4.6 show the effectiveness of our approach for the illustrative example application provided by the OPF problem.

Our future work aims to develop alternative output functions and sample selection methods in order to further improve CLA accuracy. Furthermore, we will focus on applications of the CLA methods to power system planning and resiliency problems formulated as, for instance, bilevel programs. The following chapter provides an example of one such application.

### 2. Locating Sensors and Selecting Alarm Thresholds to Identify Violations of Voltage Limits in Distribution Systems

#### 2.1 Introduction

Sensors are used in power grids for the purpose of monitoring, control, and protection. Variability in the power injections due to fluctuations in load demands and the outputs of distributed energy resources (DERs) can potentially cause violations of voltage limits, i.e., voltages outside of the bounds imposed in the ANSI C84.1 standard. Distribution system operators must identify when power injection fluctuations are causing voltage limit violations. To do so, system operators place sensors within the distribution system that measure and communicate the voltage magnitudes at their locations. For given ranges of power injection variability, we formulate and solve a *sensor placement* problem which seeks to locate the minimum number of sensors and the corresponding sensor alarm thresholds necessary to reliably identify all possible violations of voltage limit violation. Note that this problem *does not* attempt to ensure full observability of the distribution system. Rather, we seek to locate (a potentially smaller number of) sensors that can nevertheless identify all voltage limit violations space.

#### 2.2 Sensor Placement Problem

This section describes the sensor placement problem formulation by introducing notation, presenting the bilevel programming formulation that is the focus of this chapter, and detailing the objective function that simultaneously seeks to minimize the number of sensors needed and reduce the number of false positive alarms. To illustrate our proposed approach, we focus on identifying potential violations of voltage magnitude limits and we consider power injection fluctuations at each bus that lie within specified bounds. To describe the main concepts in this chapter, we consider on a balanced single-phase equivalent network representation rather than introducing the additional notation and complexity needed to model an unbalanced three-phase network representation. Extensions to consider other limits, such as restrictions on line flows, alternative characterization of power injection ranges, such as budget uncertainty sets, and unbalanced three-phase network models impose limited additional complexity to the problem formulation.

#### 2.2.1 Notation

Consider an *n*-bus power system. The sets of buses and lines are denoted as  $\mathscr{N} = \{1, \ldots, n\}$  and  $\mathscr{L}$ , respectively. One bus in the system is specified as the slack bus, where the voltage magnitude and angle is set to be  $1\angle 0^\circ$  per unit. For the sake of simplicity for the presentation of our formulation, we model each of the remaining buses as a PQ bus with particular values for the active (*P*) and

reactive (Q) power. Extensions to consider PV buses, which have particular values for the active power (P) and the voltage magnitude (V), are straightforward. (The controlled voltage magnitudes at PV buses imply that voltage violations cannot occur at these buses so long as the voltage magnitude setpoints are within the voltage limits.) The set of neighboring buses to bus *i* is defined as  $\mathcal{N}_i := \{k \mid (i,k) \in \mathcal{L}\}$ . Subscript  $(\cdot)_i$  denotes a quantity at bus *i*, and subscript  $(\cdot)_{ik}$  denotes a quantity associated with the line from buses *i* to bus *k*, unless otherwise stated. Conductance (susceptance) is denoted as G(B) as the real (imaginary) part of the admittance.

#### 2.2.2 Bilevel optimization formulation

The sensor placement formulation with the description for each variable are shown below. Bold quantities denote vectors.

$S_i$	: Sensor placement, 1 if bus <i>i</i> has a sensor, 0 otherwise.
Ci	: The cost of installing a sensor at bus <i>i</i> .
$V_i(\widetilde{V}_i)$	: Lower (upper) limit sensor threshold at bus <i>i</i> .
$\theta_{jk}$	: The voltage angle difference between buses j and k (i.e., $\theta_j - \theta_k$ ).
$P_{j}\left(Q_{j} ight)$	: Net active (reactive) power injections at bus <i>j</i> .
$\mathcal{N}_{PQ}$	: The set of all nonslack buses.

The main goal of this problem is to find sensor location(s) such that sensor(s) can capture all possible voltage violations. We formulated this problem as a bilevel optimization with the following upper-level and lower-level problems.

- Upper level: Determine sensor locations and alarm thresholds given that all extreme achievable voltages at the sensor locations are within pre-specified safety limits.
- Lower level: Find the most extreme achievable voltages given the sensor alarm thresholds and locations and the range of power injection variability.

In other words, the upper-level problem determines the sensor locations and sensor alarm thresholds such that the most extreme achievable voltages at locations without sensors are within the voltage limits. The extreme achievable voltages are provided by lower-level problems associated with each bus that maximize and minimize the voltage magnitudes over the specified range of power injections, given that the voltages at locations with sensors are within their sensor alarm thresholds (from the upper-level problem). Thus, the sensor locations and alarm thresholds output from the upper-level problem are input to the lower-level problem, and the extreme achievable voltage magnitudes output from the lower-level problem are inputs for the upper-level problem. Accordingly, the upper-level and lower-level problems must be solved simultaneously in the following bilevel optimization formulation:

$$\min_{\mathbf{s}, \widetilde{\mathbf{V}}, \mathbf{V}} c(\mathbf{s}, \widetilde{\mathbf{V}}, \mathbf{V})$$
(2.1a)

s.t. 
$$\underline{V}_i \ge V_i^{\min}, \ \overline{V}_i \le V_i^{\max},$$
 (2.1b)

$$\underbrace{U}_{i} = \underbrace{V}_{i} s_{i}, \quad \widetilde{U}_{i} = \widetilde{V}_{i} s_{i} + M(1 - s_{i}),$$
(2.1c)

$$\forall i \in \mathscr{N}_{PQ},$$

$$\underline{V}_i = \underset{V_i}{\operatorname{argmin}} V_i \qquad (\overline{V}_i = \underset{V_i}{\operatorname{argmax}} V_i), \tag{2.1d}$$

s.t. 
$$P_j = V_j \sum_{k \in \mathcal{N}_j} V_k(G_{jk} \cos(\theta_{jk}) + B_{jk} \sin(\theta_{jk})),$$
 (2.1e)

$$Q_j = V_j \sum_{k \in \mathcal{N}_j} V_k(G_{jk} \cos(\theta_{jk}) - B_{jk} \sin(\theta_{jk}), \qquad (2.1f)$$

$$\theta_1 = 0, \tag{2.1g}$$

$$\underbrace{U}_{j} \le V_{j} \le \widetilde{U}_{j}, \tag{2.1h}$$

$$P_j^{\min} \le P_j \le P_j^{\max},\tag{2.1i}$$

$$Q_j^{\min} \le Q_j \le Q_j^{\max}, \tag{2.1j}$$

$$\forall j \in \mathscr{N}_{PQ},$$

where s is a vector of sensor locations modeled as binary variables;  $\tilde{V}$  and V denote upper and lower sensor alarm thresholds, respectively;  $\widetilde{U}$  and U denote the upper and lower sensor alarm thresholds, respectively, that are enforced in the lower-level problem via the big-M formulation in (2.1c) (M is a sufficiently large constant) based on whether a sensor is actually placed at the corresponding location;  $P_i$  and  $Q_j$  denote the active and reactive power injections at bus j within a particular lower-level problem; and superscripts max and min denote upper and lower limits, respectively, on the corresponding quantity. For the upper-level problem, the objective function in (2.1a) minimizes a cost function  $c(\mathbf{s}, \mathbf{V}, \mathbf{V})$  associated with the sensor locations s and alarm thresholds V, V while ensuring that the extreme achievable voltage magnitudes calculated in the lower-level problem (i.e.,  $\overline{\mathbf{V}}$  and  $\mathbf{V}$ ) are within safety limits as shown in (2.1b). The cost function  $c(\mathbf{s}, \mathbf{V}, \mathbf{V})$  will be detailed in the following subsection. In the lower-level problem, the objective function (2.1d) computes the maximum or minimum voltage magnitudes for each PQ bus  $i \in \mathcal{N}_{PO}$ . For each lower-level problem, constraints (2.1e)–(2.1f) are the power flow equations at each bus j, constraint (2.1h) enforces the voltage magnitudes to be within voltage alarm thresholds (if a sensor is placed at the corresponding bus), and constraints (2.1i)–(2.1j) model the range of variability in the net power injections. Constraint (2.1g) sets the angle reference for the power flow equations.

#### 2.2.3 Cost function

Overly restrictive sensor thresholds can potentially trigger an alarm even when there are no voltage violations actually occurring in the system, i.e., a *false positive* or *spurious* alarm. To prevent

unnecessarily restrictive alarm thresholds, our cost function,  $c(\mathbf{s}, V, \tilde{V})$ , is defined as:

$$c(\mathbf{s}, \underline{V}, \widetilde{V}) = \sum_{i \in \mathscr{N}} c_i(s_i, \underline{V}_i, \widetilde{V}_i)$$
(2.2)

where

$$c_i(s_i, \underbrace{V}_i, \widetilde{V}_i) = \begin{cases} V_i - \widetilde{V}_i, & s_i = 1, \\ \widetilde{\delta}, & s_i = 0, \end{cases}$$
(2.3)

where  $\delta$  is a pre-specified benefit of avoiding the placement of a sensor. When  $s_i = 1$ , the objective function in (2.3) seeks to reduce the restrictiveness of the sensors in order to have fewer false positives, i.e., fewer spurious alarms. Changing the value of  $\delta$  in (2.3) models the tradeoff between placing an additional sensor and making other sensor ranges more restrictive, potentially resulting in more false positives.

#### 2.3 Reformulations of the Sensor Location Problem

The bilevel problem (2.1) is computationally challenging due to the non-convexity in the lowerlevel problem induced by the AC power flow equations in (2.1e)–(2.1f). Moreover, as we will show numerically in Section 2.4, traditional methods for reformulating the bilevel problem into a single-level problem suitable for standard solvers using the Karush-Kuhn-Tucker (KKT) conditions turn out to yield computationally burdensome problems. In this section, we provide methods for obtaining a tractable version of the bilevel sensor placement problem. We first use recently proposed "conservative linear approximations" (CLAs) of the power flow equations to convert the lower-level problem to a more tractable linear programming formulation that nevertheless retains characteristics from the nonlinear AC power flow equations. We then use various additional reformulation techniques that yield significantly more tractable problems than standard KKT-based reformulations. These reformulations first yield a (single-level) mixed-integer bilinear programming formulation that can be solved using commercial mixed-integer programming solvers like Gurobi. Via further reformulations that discretize the sensor alarm thresholds, we transform the bilinear terms to obtain a mixed-integer linear programming (MILP) problem that has further computational advantages and is suitable for a broader range of solvers.

#### 2.3.1 Conservative linear power flow approximations

The nonlinearity of the power flow equations in the lower-level problem introduces computational challenges. To tackle these challenges, we utilize a linear approximation of the power flow equations that is adaptive (i.e., tailored to a specific system and a range of load variability) and conservative (i.e., over- or under- estimate a quantity of interest to avoid constraint violations). These linear approximations are called *conservative linear approximations* (CLAs) and were first proposed in [14]; see also Chapter 1 for further details.

We use conservative linear approximations that linearly relate the voltage magnitudes at a particular bus to the power injections at all PQ buses. An example of an overestimating CLA of the voltage magnitude at bus i is the expression

$$a_{i,0} + \mathbf{a}_{i,1}^T \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}$$
(2.4)

such that the following relationship is satisfied for some specified range of power injections **P** and **Q**:

$$V_i - \left(a_{i,0} + \mathbf{a}_{i,1}^T \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}\right) \le 0.$$
(2.5)

We replace the AC power flow equations (and the angle reference) in (2.1e)–(2.1g) with a CLA as in (2.5) for all  $i \in \mathcal{N}_{PQ}$ . The bilevel problem in (2.1) then becomes:

$$\min_{\mathbf{s},\,\mathbf{\tilde{V}},\,\mathbf{\tilde{V}}} c(\mathbf{s},\mathbf{\tilde{V}},\,\mathbf{\tilde{V}})$$
(2.6a)

s.t. 
$$(\forall i \in \mathcal{N}_{PQ}),$$
  
 $V_i > V_i^{\min}, \overline{V}_i < V_i^{\max},$  (2.6b)

$$\underbrace{U}_{i} = \underbrace{V}_{i} s_{i}, \quad \widetilde{U}_{i} = \widetilde{V}_{i} s_{i} + M(1 - s_{i}), \quad (2.6c)$$

$$\underline{V}_{i} = \min_{\mathbf{P},\mathbf{Q}} \quad \underline{a}_{i,0} + \underline{\mathbf{a}}_{i,1}^{T} \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}^{i}, \qquad (2.6d)$$

$$(\overline{V}_{i} = \max_{\mathbf{P},\mathbf{Q}} \quad \overline{a}_{i,0} + \overline{\mathbf{a}}_{i,1}^{T} \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}^{i}),$$
  
s.t. 
$$\overline{a}_{j,0} + \overline{\mathbf{a}}_{j,1}^{T} \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}^{i} \ge \underline{U}_{j}, \forall j \in N_{PQ} \setminus \{i\},$$
 (2.6e)

$$\underline{a}_{j,0} + \underline{\mathbf{a}}_{j,1}^{T} \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}^{i} \leq \widetilde{U}_{j}, \,\forall j \in N_{PQ} \setminus \{i\},$$
(2.6f)

$$\begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}^{\min} \leq \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}^{i} \leq \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}^{\max}.$$
 (2.6g)

In (2.6), superscripts *i* denote quantities associated with the *i*<sup>th</sup> lower-level problem. Substituting conservative linear approximations for the AC power flow equations yields a linear programming formulation for the lower-level problem rather than the non-convex lower-level problem in (2.1). Comparing (2.6d)–(2.6g) and (2.1d)–(2.1j), we see that satisfaction of (2.6e)–(2.6f) is sufficient to ensure satisfaction of (2.1b), assuming that the conservative linear approximations do indeed reliably overestimate and underestimate the voltage magnitudes. Thus, the resulting optimization problems are sufficient to ensure that the sensor locations and alarm thresholds will identify all potential voltage limit violations. As a result, solving the reformulation (2.6) will compute sensor locations and thresholds that avoid *false negatives*, i.e., alarms will always be raised if there are indeed violations of the voltage limits. The computational advantages provided by linearity of the reformulated lower-level problem come with the potential tradeoff of additional false positives, i.e., spurious alarms. Various methods proposed in [14] for improving the accuracy of the CLAs

naturally yield variants of (2.6) that reduce the number of false positives. Additionally, in Section 2.3.6, we describe a method for post-processing the sensor alarm thresholds to further reduce the occurrences of false positives.

With a linear lower-level problem, we can apply standard techniques for reformulating the bilevel problem (2.6) as a (single-level) mixed-integer linear programming formulation. These techniques replace the lower-level problem (2.6d)–(2.6g) with its KKT conditions that are both necessary and sufficient for optimality of this problem [15] and also apply McCormick envelopes [16] to convert the bilinear product of the continuous and discrete variables in (2.1c) to an equivalent linear form. The resulting single-level problem still involves bilinear constraints associated with the complementarity conditions. These bilinear constraints are traditionally reformulated using binary variables. Commercial MILP solvers are applicable to this traditional reformulation approach, which we denote throughout this chapter as the "KKT formulation" and present in more detail in Appendix 2-A. However, the KKT formulation is significantly less scalable than the reformulations we will present next, limiting its scalability to small test cases.

#### 2.3.2 Duality of the lower-level problem

As an alternative to the KKT formulation, one can reformulate a bilevel problem into a singlelevel problem by *dualizing* the lower-level problem. This technique can only be usefully applied to problems with specific structure where the optimal objective value of the lower-level problem is a constraint of the upper-level problem. In this special case, we can significantly improve tractability compared to the KKT formulation and, as we will discuss and demonstrate later, consider larger systems with multiple switching configurations.

To illustrate this refomulation, we present a simpler problem setting where we assume that there are no violations of upper voltage limits (i.e., that the maximum achievable voltages are always within safety limits without the need for any sensors). Thus, the maximization problem in (2.6d) and the upper limit thresholds in (2.6f) are ignored. Generalizations to consider both lower and upper voltage limits are straightforward.

Let  $y^i$  be the vector of dual variables associated with lower-level problem *i* and *I* be an identity matrix with an appropriate size. By dualizing the lower-level problem, we obtain:

$$\min_{\mathbf{s},\,\mathbf{V}} \quad c(\mathbf{s},\mathbf{V}) \tag{2.7a}$$

s.t. 
$$(\forall i \in \mathscr{N}_{PQ}),$$

$$\underline{V}_i \ge V^{\min}, \tag{2.7b}$$

$$\underbrace{U}_{i} = \underbrace{V}_{i} s_{i}, \qquad (2.7c)$$

$$\underline{V}_{i} = \max_{\mathbf{y}^{i}} v_{s}^{T} \mathbf{y}^{i} + \underline{a}_{i,0}, \qquad (2.7d)$$

s.t. 
$$A\mathbf{y}^i = \underline{\mathbf{a}}_{i,1},$$
 (2.7e)

$$\mathbf{y}^i \ge \mathbf{0},\tag{2.7f}$$

where

$$A^{T} = \begin{pmatrix} -I \\ I \\ \overline{\mathbf{a}}_{1,1}^{T} \\ \vdots \\ \overline{\mathbf{a}}_{n,1}^{T} \end{pmatrix}, \\ v_{s} = \begin{pmatrix} -\mathbf{P}^{\max} \\ -\mathbf{Q}^{\max} \\ \mathbf{P}^{\min} \\ \mathbf{Q}^{\min} \\ U_{1} - \overline{a}_{1,0} \\ \vdots \\ U_{n} - \overline{a}_{n,0} \end{pmatrix}.$$

Due to strong duality of the linear lower-level problem, the reformulation (2.7) is equivalent to (2.6) and does not directly provide any advantages. However, there is special structure in our bilevel optimization formulation that we can now exploit to dramatically improve tractability. Specifically, outputs from each lower-level problem (2.7d)–(2.7f) only appear in a single inequality constraint (2.7b). This means that we do not need to compute the maximum value of the lower-level problem (2.7d)–(2.7f), but rather *only need to show that there exists some choice of dual variables*  $\mathbf{y}^i$  for *which* (2.7b) *is feasible*. This allows us to replace (2.7b) and (2.7d) with the following single-level formulation:

$$\min_{\mathbf{s},\mathbf{V}} \quad c(\mathbf{s},\mathbf{V}) \tag{2.8a}$$

s.t. 
$$(\forall i \in \mathcal{N}_{PQ}),$$
  
 $U_i = V_i s_i$  (2.8b)

$$\sum_{i} V_{i}^{T} v_{i}^{i} + q_{i} > V^{\min}$$
(2.60)

$$\mathbf{v}_{s} \mathbf{y} + \underline{u}_{i,0} \leq \mathbf{v}_{i} \quad , \tag{2.80}$$

$$\mathbf{A} \mathbf{y}^{i} = \underline{\mathbf{a}}_{i,1}, \tag{2.8d}$$

$$\mathbf{y}^{\iota} \ge \mathbf{0}. \tag{2.8e}$$

We refer to (2.8) as the "bilinear" formulation due to the bilinear product of the sensor threshold variables U and the dual variables  $y^i$  in (2.8d). Similar to the KKT reformulation discussed previously, problem (2.8) is a single-level optimization problem. However, problem (2.8) has the major advantage that no additional binary variables are required (beyond those associated with the sensor locations in the upper-level problem) since there is no analogous equations to the complementarity condition. Our numerical results in Section 2.4 show that modern mixed-integer programming

solvers like Gurobi are capable of solving problem (2.8) for much larger systems than are possible with the standard KKT reformulation. However, the bilinear nonlinearity in (2.7d) induces computational challenges that can be addressed with one additional reformulation that is discussed next.

#### 2.3.3 Bilinear to mixed-integer linear programming

The optimization problem in (2.8) is nonlinear due to bilinear terms in (2.7d). To address this nonlinearity, we discretize the sensor thresholds, which are continuous variables in the previously presented formulations, into *d* discrete steps with size  $\varepsilon$ . Specifically, we define the vector of sensor thresholds at each bus, *V*, as

$$\underbrace{V}_{i} = \eta^{T} v_{t}, \qquad (2.9)$$

where

$$\eta = \begin{bmatrix} \eta_{0,1} & \eta_{0,2} & \cdots & \eta_{0,n} \\ \eta_{1,1} & \eta_{1,2} & \cdots & \eta_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{d,1} & \eta_{d,2} & \cdots & \eta_{d,n} \end{bmatrix}, \qquad (2.10a)$$

$$v_t = \begin{bmatrix} 0 \\ V^{\min} \\ V^{\min} + \varepsilon \\ V^{\min} + 2\varepsilon \\ \vdots \\ V^{\min} + (d-1)\varepsilon \end{bmatrix}, \qquad (2.10b)$$

$$\sum_{i=1}^d \eta_{i,k} = 1, \qquad (2.10c)$$

for all  $k \in \mathcal{N}_{PQ}$ . Note that this discretization exploits the fact that any sensor alarm threshold will necessarily be above the lower voltage limit  $V_i^{\min}$ . Using this discretization, the constraints (2.8c) now contain bilinear products of binary variables. These products can be equivalently transformed into a mixed-integer *linear* (as opposed to bilinear) programming formulation using McCormick Envelopes [16]. With McCormick Envelopes and discrete sensor thresholds, the problem (2.8) thus becomes a MILP that can be computed using any MILP solver. We refer to the reformulation of (2.8) using the sensor threshold discretization (2.9) as the "MILP" formulation.

To further improve tractability, we can remove unnecessary binary variables by inspecting data from the samples of power injections used to compute the conservative linear approximations of the power flow equations. Let *m* be a bus where its voltage magnitude never reaches the highest discretized sensor threshold value (i.e.,  $V^{\min} + (d-1)\varepsilon$ ) in any of the sampled power injections. Given a sufficiently comprehensive sampling of the range of possible power injections, we can then

simplify the discretized representation of the sensor alarm threshold as:

$$V_m = 0 \cdot \eta_{0,m} + V^{\min} \cdot \eta_{1,m}, \qquad (2.11a)$$

$$\eta_{0,m} + \eta_{1,m} = 1. \tag{2.11b}$$

In other words, we can perform a pre-screening to eliminate binary variables associated with sensor thresholds at buses that will never violate their voltage limits. We henceforth call this data-driven simplification technique "binary variable removal" (BVR).

#### 2.3.4 A heuristic technique for benchmarking purposes

To benchmark the performance of our proposed sensor placement problem formulation, we will use a heuristic technique that exploits the traditional behavior of distribution systems where voltage magnitudes typically drop as one moves down a feeder away from the substation. This behavior suggests that the ends of each feeder may be good locations for locating sensors since the voltage magnitude limits are likely to be violated here first.

This heuristic technique avoids the computational burden from computing the conservative linear power flow approximations and solving some reformulation of the bilevel optimization problem. However, this heuristic does not always work as it can introduce false negatives (i.e., failure to alarm when there is a voltage limit violation) and, moreover, the results from this heuristic are not optimal. Some system characteristics where this heuristic technique may perform poorly include systems that have (i) DERs at or near the ends of some branches, (ii) the ability to reconfigure the network by opening and closing switches, and (iii) many branches that would each require sensors. With these system characteristics, the lowest voltage in each branch may no longer at the end of that branch. Moreover, placing sensors at the end of all branches can be very costly in systems that are complicated with many branches. Illustrative examples of systems with such characteristics are provided in Section 2.4.4.

#### 2.3.5 Comparisons of each formulation

The previous subsections present several problem reformulations that convert the bilevel sensor placement problem (2.1) into various single-level problems that can be solved with mixed-integer programming solvers like Gurobi. Each reformulation has different computational characteristics and yields solutions with differing accuracy. We next compare the KKT-based reformulation (2.14) described in Appendix 2-A with the duality-based bilinear reformulation (2.8) described in Section 2.3.2 according to the following characteristics:

- 1. The presence of nonlinear terms.
- 2. The number of decision variables.
- 3. The number of constraints.

Both the KKT and bilinear reformulations involve nonlinear functions. The KKT reformulation

introduces nonlinearities associated with the complementary slackness condition, which, in our formulation, is the product of a dual variable and a linear function of the primal variables. This nonlinearity can be linearized by introducing additional binary variables and big-M constraints as described in Appendix 2-A. The duality reformulation has bilinear terms in (2.7d). These bilinear terms can be linearized by discretizing the sensor thresholds using binary variables as described in Section 2.3.3.

To assess the number of decision variables and the number of constraints, consider a system where there are *b* PQ buses, of which there are *r* buses where the voltage magnitudes may violate their limits. (Recall that we employ a pre-screening method to identify buses where the voltage limits might be violated based on the sampled power injections used to compute the conservative linear approximations of the power flow equations as discussed in Section 2.3.3.) The total number of decision variables in the KKT reformulation is  $5 \cdot b \cdot r + 2 \cdot b$  ( $2 \cdot b \cdot r$  from power injections,  $3 \cdot b \cdot r$ from dual variables, *b* from the voltage thresholds, and *b* from the sensor locations). Our proposed duality-based bilinear reformulation involves only  $3 \cdot b \cdot r + 2 \cdot b$  decision variables. All of these variables are similar to the KKT reformulation except for the power injections, which are entirely removed by duality.

Regarding the number of constraints, the duality-based bilinear reformulation does not have the stationarity conditions, primal feasibility, or power injections directly involved, resulting in a reduction of  $2 \cdot b \cdot r + b \cdot r + 2 \cdot b \cdot r = 5 \cdot b \cdot r$  constraints. The implications that these differences have on tractability is best assessed via the solution times presented in section 2.4.

#### 2.3.6 Approximate gradient descent

Solving any of the reformulated bilevel optimization problems may lead to *false positives*, i.e., alarms that occur in the absence of a voltage limit violation. This is an inherent result of the conservativeness of our conservative linear approximations that replace the power flow equations in the lower-level problem (2.1e)–(2.1g). These false positives are undesirable because they may lead system operators to take unnecessary corrective actions that could reduce the efficiency of system operations or put avoidable wear on system components.

To reduce the number of false positives, this section proposes a post-processing step that iteratively adjusts the sensor thresholds that result from the reformulated bilevel optimization problems. We refer to this post-processing step as the Approximate Gradient Descent (AGD) method. This method uses the voltage magnitudes associated with a large number of sampled power injections within the considered range of power injection variability. These samples can be the same as those used to compute the conservative linear approximations of the power flow equations.

In this section, let superscript k denote the  $k^{\text{th}}$  iteration of the algorithm. Let  $\varepsilon_{AGD}$  be a step size for adjusting the sensor thresholds and  $f^k$  be the number of false positives (spurious alarms) from the sampled power injections using the sensor thresholds in the  $k^{\text{th}}$  iteration. Using the sampled power injections, this method computes an "approximate gradient" indicating how small changes



Figure 2.1: A diagram showing the process of computing sensor locations and alarm thresholds. The final step is performed iteratively until any additional changes would result in false negatives (i.e., undetected constraint violations).

to the sensor alarm thresholds affects the number of false positives. The approximate gradient at iteration k is denoted as  $g^k$ . We denote the set of buses with sensors as  $\mathcal{N}_s$ . Subscripts denote the bus number and superscripts denote the iteration number.

Similar to the presentation of the reformulations above, we focus on the lower voltage limits for ease of presentation. Extensions to consider upper voltage limits are straightforward. Let  $\Delta f_i^k$  represent the change in the number of false positives among the sampled power injections using the sensor thresholds in the  $k^{\text{th}}$  iteration when the sensor alarm threshold  $V_i^k$  is reduced by  $\varepsilon_{AGD}$  (leaving all other sensor thresholds unchanged). We then compute an approximate gradient  $g^k$  by comparing the values of  $\Delta f_i^k$  across different buses *i*:

$$g^{k} = \frac{\Delta f^{k}}{\sum_{i \in \mathcal{N}_{s}} \Delta f_{i}^{k}}.$$
(2.12)

The approximate gradient  $g^k$  thus points in the direction of the steepest reduction in the (empirically determined) number of false positives. Each iteration of the gradient descent method updates the sensor thresholds according to the update rule:

$$\underline{V}^{k+1} = \underline{V}^k + \varepsilon_{AGD} \cdot g^k. \tag{2.13}$$

The AGD method terminates when taking an additional step would result in the appearance of false *negatives*, i.e., undetected violations of voltage limits. Fig. 2.1 illustrates the overall process for computing the sensor locations and alarm thresholds starting from computing the conservative linear approximations of the power flow equations (a pre-processing step), solving a reformulation of the bilevel problem, and adjusting thresholds via the approximate gradient descent method (a post-processing step).

#### 2.3.7 A system with multiple possible configurations

Sensors are placed once and operate for extended periods of time during which network conditions may change. The range of power injection variability specified in the bilevel problem formulation (2.1i)–(2.1j) models changes due to, e.g., time-varying load demands and fluctuating power

outputs of distributed energy resources. The topology of electric distribution networks may also change due to network reconfiguration as system operators adjust the status of switches. Network reconfiguration is particularly relevant in the sensor placement problem as the relationships between sensors within the distribution system can effectively change with adjustments to the network topology, even though the sensor is physically located at the same bus.

To address this, our sensor placement formulation can be extended to consider a set of possible network topologies. In this context, the goal of the sensor placement problem is to determine locations for sensors and sensor alarm thresholds that will reliably identify voltage limit violations for any topology within the considered set. In this setting, the sensor locations must be selected once and remain consistent across all topologies, but the sensor alarm thresholds may vary between different topologies. With *m* different topologies for an *n*-bus system, the computational complexity is similar to solving a sensor placement problem for an  $(n \cdot m)$ -bus system with one configuration.

For each topology, we compute different conservative linear approximations of the power flow equations and introduce additional lower-level problems (i.e., for a specific bus, each configuration introduces different constraints (2.8b)-(2.8e)). We emphasize that the sensor locations are the same across all topologies, which is accomplished by having a single binary variable indicating whether a sensor is located at a particular bus, not indexed by the network topology.

#### 2.4 Numerical Tests

In this section, we perform numerical experiments on a number of test cases to analyze the sensor locations and thresholds, demonstrate the advantages of our problem reformulations and the post-processing step, and compare results and computational efficiency from different problem formulations.

The test cases we use in these tests are the 10-bus system *case10ba*, the 33-bus system *case33bw*, and the 141-bus system *case141* from MATPOWER [13]. For the CLAs, we minimize the  $\ell_1$  error with 1000 samples in the first iteration and 4000 additional samples in a sample selection step and choose a quadratic output function of voltage magnitude ( $V^2$ ). (See [14] and Chapter 1 for a discussion on computationally efficient iterative methods for computing CLAs and variants of CLAs that approximate different quantities in order to improve their accuracy.) All power injections vary within 50% to 150% of the load demand values given in the MATPOWER files except for *case33bw* where we consider a variant with solar panels installed at buses 18 and 33. The loads at these two buses vary within -200% to 150% of the given values.

We implement the single-level reformulations of the sensor placement problem in MATLAB using the YALMIP toolbox [17] and use Gurobi as a solver [18] with a tolerance of a 0.005 gap value (also known as MIPgap in Gurobi) unless otherwise stated. The step size for AGD is  $\varepsilon_{AGD} = 2 \times 10^{-4}$  per unit. The computation times for the CLAs, the voltage limits, and the value of  $\delta$  in the objective function (2.3) for all test cases are provided in Table 2.7 in Appendix 2-B.

	K	KT§		Bilinear		1	AILP w/o	BVR		MILP													
	cas	se10ba	case10ba	case33bw	case141	case10ba	case33bw	case141	case10ba	case33bw	case141												
Running Optima	lity 2	26.7	1.96	4.47	46.52	1.95	3.19	69.6	1.54	2.87	22.95												
time [s] AGE	)	—	0.66	7.12	95.3	—	10.6	69.9	—	7.75	49.7												
Sensor location(	s)	10	10	12, 16, 32	79, 80, 82, 85	10	14, 31	76, 80, 86	10	14, 30	80, 86												
Sansor thrashold	(0)	0.9	0.0	0.0	0.0	0.9	0.0	0.9	0.0	0.0	0.0	0.0	0.9	0.0	0.0017	0.9118, 0.9167,	0.92, 0.9213,	0.0	0.919	0.92, 0.9295,	0.0	0.9195	0.929
Sensor uneshold	(8)		0.9017	0.9113	0.93, 0.93	0.9	0.9195	0.9295	0.9	0.9190	0.9295												
with AGD					0.0	0.9118, 0.9144,	0.92, 0.9213,		0.9169	0.92, 0.9216,		0.917	0.9212										
with AOD		_	0.9	0.9105	0.9217, 0.9201	_	0.9148	0.9202		0.9171	0.9201												
# feasible point	s 7	7317	7317	9560	9955	7317	9560	9955	7317	9560	9955												
# false positive(	s)	0	401	674	7207	0	2131	6664	0	1033	6664												
with AGD		_	0	209	4		556	11		339	6												
# false negative	s	0	0	0	0	0	0	0	0	0	0												

Table 2.1: Results showing sensor alarm thresholds and number of false positives from KKT, bilinear, and MILP reformulations.

<sup>8</sup>The solver takes over 55000 s to compute the optimal solution but fails to find one for *case33bw* and *case141*.



Figure 2.2: Bar graphs showing the computation times for solving various problem reformulations and executing the approximate gradient descent method.

#### 2.4.1 Sensor locations

Section 2.3.5 discusses the number of decision variables in each reformulation. The difference in the number of decision variables is one of the main factors that directly impacts the solver's computation time. One of the techniques that we use is removing unnecessary binary variables in the MILP formulation based on the observation from data obtained in the pre-processing step (i.e., computing CLAs). Another factor is the presence of nonlinear expressions in each formulation. The impact of these factors is difficult to characterize analytically, so we next present empirical comparisons on the three aforementioned test cases. We compare the quality of results and computation time from the following reformulations: (i) the KKT reformulation (2.14), (ii) the duality-based bilinear problem (2.8), (iii) the MILP reformulation that uses discretized sensor alarm thresholds (2.9) without applying the binary variable removal (BVP) method and (iv) the MILP reformulation that uses discretized sensor alarm thresholds (2.9) in combination with the BVR method.

The first test case is the 10-bus system case10ba, which has a simple single-branch distribution

network. We consider a variant of this case where the loads are 60% of the given values in the MATPOWER file. The results from each reformulation place a sensor at the end of the branch (i.e., the furthest bus from the substation) with an alarm threshold of 0.9 per unit (at the voltage limit). Fig. 2.2a compares computation time from the four reformulations. The reformulation using the KKT conditions takes 26.7 s while the bilinear, the MILP without BVR, and the MILP formulations take only 1.96, 1.95, and 1.54 s, respectively. Since the sensor threshold value for the KKT and both MILP formulations is at the safety limit, AGD is not needed. On the other hand, the bilinear formulation gives a higher alarm threshold. As a result, the AGD method is applied as the postprocessing step to achieve the lowest possible threshold without introducing false alarms. The number of false positives reduces from 0.055% to 0%. The run time of the AGD method is 0.66s.

The second test case is the 33-bus test system *case33bw*, which has multiple branches. Table 2.1 shows the run time of the bilinear and the two MILP formulations. We exclude the run time for the KKT formulation since the solver fails to find even a feasible (but potentially suboptimal) point within a time limit of 55000s (15 hours). Our final test case is the 141-bus system *case141*. Similar to the 33-bus system, the solver could not find the optimal solution for the KKT formulation within a time limit of 55000s (15 hours). Table 2.1 again shows the results for this test case, and Figs. 2.2b and 2.2c compares the computation times for the bilinear and MILP reformulations.

To assess the computational speed and solution quality from each reformulation, Table 2.1 shows both the computation times and the results of randomly drawing sampled power injections within the specified operating range, computing the associated voltages by solving the power flow equations, and finding the number of false positive alarms (i.e., a voltage at a bus with a sensor is below a sensor threshold but there is no voltage violation happening in the system). The results for the 33-bus and 141-bus test cases given in Table 2.1 illustrate the performance of the proposed reformulations. Whereas the traditional KKT-based approach is computationally intractable, our proposed reformulations find solutions within approximately one minute, with the MILP reformulation with the BVR method typically exhibiting the fastest performance. (We further note that the MILP reformulation has advantages in the range of solvers that can handle such problems relative to the more restrictive set of solvers that suitable for the bilinear reformulation.) The solutions to the reformulated problems place a small number of sensors (two to four sensors in systems with an order of magnitude or more buses). The solutions have a similar pattern: in all cases, there are no false negatives. In other words, as intended, the sensor alarms for every sample where the voltage limits are violated. Conversely, there are a relatively large number of false positives (i.e., spurious sensor alarms) prior to applying the AGD method. After applying the AGD method, the number of false positives decreases dramatically to a small fraction of the total number of samples. Thus, the overall sensor placement approach is a computationally efficient method for identifying a small number of sensor locations and associated alarm thresholds that reliably identify voltage constraint violations with no false negatives (missed alarms) and few false positives (spurious alarms).

		Bili	near	MI	LP
MIP Gap Tolerance		0.3%	0.1%	0.3%	0.1%
Running	Optimality	4.81	5.51	3.01	7.35
time [s]	AGD	4.82	4.59	8.21	5.14
Sensor location(s)		15, 31	18, 31	14, 32	17, 33
Company through all d(a)		0.9169	0.9136	0.9195	0.9125
Selisor u	lifeshold(s)	0.9126	0.9126	0.918	0.912
		0.9154	0.9123	0.9169	0.9111
with	IAOD	0.9117	0.9117	0.9136	0.9106
# feasil	ble points	9560	9560	9560	9560
# false	positive(s)	516	537	1941	334
with AGD		195	260	495	74
# false negatives		0	0	0	0

Table 2.2: Results from different MIP gap tolerances for case33bw

#### 2.4.2 MIP gap tolerance, solver times, and solution quality

Each of the problem formulations involve a mix of continuous and binary variables, thus requiring solution from mixed-integer programming solvers like Gurobi. These solvers are based on branch-and-bound algorithms that iteratively update upper and lower bounds on the optimal solution, terminating when the difference between these bounds (i.e., the "MIP Gap") converges to a value less than a specified tolerance. The choice of MIP gap tolerance can directly affect the running time and the quality of the solution (e.g., sensor thresholds and number of sensors).

We investigate the effect of two different choices for the MIP gap tolerance, 0.3% and 0.1%, on the bilinear and MILP reformulations for the test cases *case33bw* and *case141*. The results are shown in Table 2.2 and Table 2.3, respectively. As expected, these results suggest that decreasing the MIP gap tolerance can increase the computational times in some cases, with approximately twice as much time required for the MILP reformulation with *case33bw* and the bilinear formulation with *case141*. Nevertheless, the computation times with the smaller MIP gaps are still reasonable in all cases (less than two minutes), and the MIP gap tolerance had little impact on the computation times for other cases. Regarding solution quality, tighter MIP gap tolerances can lead to fewer sensors (as in the bilinear reformulation with *case141*) and fewer false positives (as in the MILP reformulation with *case141*) and fewer false positives (as in the MILP reformulation with *case33bw*). However, the results are not uniformly improved by tightening the MIP gap tolerance, as the 0.1% tolerance led to the same number of sensors and *more* false positives than the 0.3% tolerance after applying the AGD method. This suggests a potential benefit of assessing the performance of multiple "nearly optimal" solutions obtained with different MIP gap tolerances.

		Bilin	MILP		
MIP Gap	o Tolerance	0.3%	0.1%	0.3%	0.1%
Running	Optimality	49.9	94.3	27.6	27.7
time [s]	AGD	81.4	65.4	45.6	45.6
Sensor	location(s)	79, 80,	79, 80,	80,	80,
3611301	location(s)	82, 85	85	86	86
		0.92, 0.9213	0.9297,	0.929,	0.929,
Selisor u		0.93, 0.93	0.92, 0.93	0.9295	0.9295
with		0.92, 0.9213	0.9219,	0.9212,	0.9214
witt	IAOD	0.9217, 0.9201	0.92, 0.9201	0.9201	0.9201
# feasil	ble points	9955	9955	9955	9955
# false positives		7207	7174	6664	6664
with AGD		2	3	5	5
# false negatives		0	0	0	0

Table 2.3: Results from different MIP gap tolerances for *case141* 

#### 2.4.3 Multiple configurations

The previous results described the sensor placements for the *case10ba*, *case33bw*, and *case141* systems in their nominal network topologies. In this section, we next demonstrate the effectiveness of our problem reformulations for variants of these systems with multiple network configurations. We consider a variant of the *case33bw* system with three different network configurations and two solar PV generators installed at buses 18 and 33. The first configuration is the nominal topology given in the MATPOWER version of the test case. In the second configuration, the line connecting buses 6 and 7 is removed and a new line connecting buses 4 and 18 is added. The third configuration removes the line connecting buses 2 and 19 and adds a new line connecting buses 19 and 25.

Table 2.4 shows the results from using the bilinear and MILP reformulations to solve the multipleconfiguration problem for this case. The results generally mirror those from the single-networkconfiguration test cases shown earlier in that computation times are still reasonable (approximately a factor of four larger), there are no false negatives (i.e., the sensors still alarm for all sampled power injections with voltage limit violations), and a small number of false positives after applying the AGD method.

Now we consider each configuration separately where sensor locations are not necessary the same. The results from Table 2.5 indicate that configurations 1 and 2 require only two sensors while configuration 3 requires only one sensor as opposed to three-sensor solution obtained from the multiple-configuration problem. This demonstrates the need to jointly consider network topologies in one problem for such situations.

		Bilinear				MILP	
		Config 1	Config 2	Config 3	Config 1	Config 2	Config 3
Running	Optimality		20.1			7.93	
time [s]	AGD	6.65	2.59	1.52	7.56	3.09	3.26
Sensor l	ocation(s)		- 8, 14, 26, 33			- 9, 14, 30	
Sensor	thresholds	0.91, 0.919,	0.9126, 0.91,	0.91, 0.91,	0.91, 0.919,	0.9185, 0.91,	0.91, 0.91,
Sciisor	uncsholds	0.91, 0.9113	0.91, 0.9111	0.9106, 0.9137	0.919	0.919	0.9165
		0.91, 0.917,	0.9118, 0.91,	0.91, 0.91,	0.91, 0.9169,	0.9167, 0.91,	0.91, 0.91,
witt	IAUD	0.91, 0.9106	0.91, 0.9111	0.9102, 0.9137	0.9173	0.919	0.9153
# feasil	ble points	9560	8292	9112	9560	8292	9112
# false positives		619	486	128	919	1238	655
with	AGD	221	176	28	347	396	292
# false	negatives	0	0	0	0	0	0

Table 2.4: Results for the 33-bus system with three network configurations.

#### 2.4.4 Comparison to a heuristic sensor placement technique

We next demonstrate the performance of the heuristic sensor placement technique described in Section 2.3.4 in the context of multiple network configurations for the three-configuration variant of the test case *case33bw*. We study the effects of two versions of this heuristic: (1) place sensors at the end of all branches based on configuration 1 and (2) place sensors at the end of branches considering all configurations. The results are shown in Table 2.6. For the first version of this heuristic, we place four sensors. This technique works well in configuration 1; however, it introduces false negatives (failure to alarm in cases with voltage limit violations in approximately 10% to 20% of the power injection samples for configurations 2 and 3 since the sensors are instead located in the middle of some branches and thus do not capture all possible violations. To reduce the number of false negatives, we next consider sensor locations based on all configurations, i.e., the second version of

Table 2.5: Results for the 33-bus system with three network configurations where sensor locations are not necessary the same.

	Config 1	Config 2	Config 3
Sensor location(s)	14, 30	9, 31	30
Sensor	0.9195	0.9185	0.0185
threshold(s)	0.9190	0.9185	0.9105
with AGD	0.9170	0.9167	0.0152
with AOD	0.9149	0.9185	0.9155
# feasible points	9560	8292	9112
# false positives	1048	1245	1453
with AGD	348	407	292
# false negatives	0	0	0

Case		Config 1	Config 2	Config 3
	Sensor location(s)	]	18, 22, 25, 3	3
n 1	Sensor thresholds	0.91	, 0.91, 0.91,	0.91 —
rsio	# feasible points	9560	8292	9112
vel	# false positives	0	0	0
	# false negatives	0	1626	885
	Sensor location(s)	7, 1	8, 22, 25, 20	5, 33 —
n 2	Sensor thresholds	0.91, 0.91	, 0.91, 0.91,	0.91, 0.91
rsio	# feasible points	9560	8292	9112
vei	# false positives	0	0	0
	# false negatives	1	0	0

Table 2.6: Results for the 33-bus system with three network configurations using two variants of the heuristic technique

the heuristic. This results in six sensors being placed and only one occurrence of a false negative. Comparing to our approach (refer to Table 2.4), we only need three sensors (as opposed to six sensors) to detect all violations. We also note that the heuristic further reduces the (already small) number of false positives (spurious sensor alarms) associated with the solution to the reformulated problem, but still has a false negative.

#### 2.5 Conclusion

This chapter has formulated a bilevel optimization problem that seeks to minimize the number of sensors needed to detect violations of voltage magnitude limits in an electric distribution system. This problem aims to determine sensor locations and alarm thresholds that result in reliably avoiding false negatives (i.e., undetected constraint violations) while reducing the number of false positives (spurious alarms when the voltages are all actually within their limits).

Since this bilevel problem is difficult to solve directly, we developed several approaches for addressing its computational challenges. We first addressed the power flow nonlinearities in the lower-level problem via previously developed conservative linear approximations of the power flow equations. These approximations provide overestimates and underestimates of the nonlinear power flow equations over the relevant operational range. By replacing the nonlinear power flow equations with these conservative linear approximations, we can ensure that the resulting sensor locations and alarm thresholds are sufficient to identify any constraint violations that might occur for any power injections within a specified range of power injection fluctuations.

Even after employing the conservative linear approximations, the sensor placement problem remains computationally challenging due to its bilevel nature. We use both analytical arguments and several test cases to show that standard techniques for reformulating the sensor placement problem to a single-level formulation via the KKT conditions of the lower-level problem yield optimization problems that are only tractable for small systems. We therefore exploit structure specific to this problem to obtain single-level mixed-integer programming formulations that can be solved with commercial solvers. Key insights in these reformulations include the importance of discretizing the sensor thresholds and a tailored dualization approach that avoids introducing unnecessary additional discrete variables, as would be required using traditional methods for bilevel problems.

The resulting sensor placement reformulations require substantially less computational time than standard reformulation techniques, thus enabling us to scale up to problems with tens to hundreds of buses. Furthermore, we developed extensions to these reformulations that consider the possibility of multiple network topologies, with the results still showing strong computational scalability characteristics.

To improve the sensor placements resulting from this bilevel problem formulation, we developed a post-processing technique that reduces the number of false alarms. Using an approximate gradient approach, this post-processing technique adjusts the sensor thresholds to make them less conservative while not introducing false negatives (missed alarms). The combination of the bilevel problem reformulation and this post-processing technique allows us to compute sensor locations and alarm thresholds that have a small number of false alarms and no missed alarms, as validated numerically using out-of-sample testing.

#### Appendix 2-A: Bilevel problem and KKT conditions

Following standard reformulation techniques [15], replacing the lower-level problem with its KKT conditions yields the following single-level problem. Similar to the presentation of the dual reformulation (2.7), we only consider violations of the lower voltage limits for the sake of simplicity in the presentation, but extensions that consider both upper and lower voltage limit violations are straightforward. The following problem setup demonstrates the sensor placement problem using the KKT conditions:

$$\min_{\mathbf{s},\mathbf{V}} \quad c(\mathbf{s},\mathbf{V}) \tag{2.14a}$$

s.t.  $(\forall i \in \mathscr{N}_{PQ}),$ 

$$\underbrace{V}_{i} \ge V_{i}^{\min}, \tag{2.14b}$$

$$\underbrace{V}_{i} = \underbrace{V}_{i} s_{i},$$
(2.14c)

$$\underline{a}_{i,0} + \underline{\mathbf{a}}_{i,1}^{T} \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}^{T} \ge V_{i}^{\min}, \qquad (2.14d)$$

$$\underline{a}_{i,1} - \sum_{k \in \mathscr{N}_{PQ}} \lambda_k^i \overline{\mathbf{a}}_{k,1} - \sum_{k \in \mathscr{N}_{PQ}} \gamma_k^j e_k + \sum_{k \in \mathscr{N}_{PQ}} \pi_k^i e_k = 0, \qquad (2.14e)$$

$$\overline{a}_{j,0} + \overline{\mathbf{a}}_{j,1}^T \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}^i \ge U_j, \,\forall j \in N_{PQ} \setminus \{i\},$$
(2.14f)

$$\begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}^{\min} \leq \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}^{t} \leq \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}^{\max}, \qquad (2.14g)$$

$$\lambda_{j}^{i} \cdot \left( \underbrace{U}_{j} - \overline{a}_{j,0} - \overline{\mathbf{a}}_{j,1}^{T} \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}^{i} \right) = 0, \ \forall j \in N_{PQ} \setminus \{i\},$$
(2.14h)

$$(\Gamma^{i})^{T} \odot \left( \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}^{\min} - \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}^{i} \right) = 0, \qquad (2.14i)$$

$$(\Pi^{i})^{T} \odot \left( \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}^{i} - \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix}^{\max} \right) = 0,$$
(2.14j)

$$\Gamma^{i}, \Pi^{i} \ge \mathbf{0}, \lambda_{j}^{i} \ge 0, \forall j \in N_{PQ} \setminus \{i\}.$$
(2.14k)

where the operator  $\odot$  is the element-wise multiplication,  $e_i$  is the  $i^{\text{th}}$  column of the identity matrix;  $\lambda$ ,  $\Gamma := (\gamma_1, \gamma_2, \dots, \gamma_{2m})$ , and  $\Pi := (\pi_1, \pi_2, \dots, \pi_{2m})$  are dual variables associated with the constraint (2.6e) (given that there are *m* PQ buses), the lower bounds on the ranges of power injections, and the upper bounds on the ranges of power injections, respectively. Equations (2.14e)–(2.14j) are the KKT conditions of the lower-level problem. Equation (2.14e) is the stationarity condition. The primal feasibility conditions in (2.14f)–(2.14g) are similar to the constraints (2.6e)–(2.6g) in the original problem with conservative linear approximations of the power flow equations. The complementary slackness conditions are (2.14h)–(2.14j) and the dual feasibility condition is (2.14k). Observe that the complementary slackness conditions give rise to nonlinear functions due to the multiplication of the dual variables  $\lambda$ ,  $\Gamma$ , and  $\Pi$  with the primal variables **P** and **Q**. To handle these nonlinearities, traditional methods for bilevel optimization replace these products using additional binary variables and a big-M reformulation which ensures that at least one of the terms in the product equals zero (i.e., a logical OR condition). To do so, traditional methods require specifying upper bounds on the dual variables  $\lambda$ ,  $\Gamma$ , and  $\Pi$ . Appropriate upper bounds are difficult to determine. Choosing too large of values for these bounds results in poor computational performance since the relaxations used by MILP solvers will be loose. Choosing too small of values for these bounds will change the problem, potentially leading to suboptimal solutions or even infeasibility of otherwise feasible problems.

#### Appendix 2-B: Running time and data

The table below shows the time required for computing the conservative linear approximations, the voltage limits, and the benefit from avoiding placing a sensor (i.e.,  $\delta$  in (2.3)) for the numeric test cases *case10ba*, *case33bw*, and *case141*. This information is used in Section 2.4.

	case10ba	case33bw	case141
CLA running	59	108	1/15
time [s]	50	190	1413
Safety limit [pu]	0.90	0.91	0.92
Benefit from avoiding	0.88	0.80	0.0
a sensor placement ( $\delta$ )	0.00	0.09	0.9

Table 2.7: CLA running time and data for test cases.

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# Part III

## Identifying Secure Operation Ranges For DER Control by Third-Party Aggregators

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#### 1. Introduction

With increasing numbers of distributed energy resources (DERs) connected at the distribution level, distribution grids are no longer passive loads in the transmission system operation. This can sometimes cause issues in distribution grid operation, as DERs change typical loading conditions and introduce power quality issues such as overvoltages or voltage unbalance. On the other hand, the increasing number of dispatchable DERs in distribution grids offers the possibility to provide services such as operating reserves to the transmission systems. One challenge associated with controlling DERs is that they are managed by different (and sometime multiple) entities such as their individual owner, the responsible distribution utility or distribution system operator (DSO), as well as third-party aggregators. The control objectives of these entities may be different and conflicting, making it important to establish effective frameworks for coordination. In this part of our project, we assume that DERs are controlled by aggregators with no knowledge of distribution grid constraints. Our goal is to establish a range of secure operating ranges for the DERs. These ranges are defined as allowable ranges of aggregate active power response from the DERs (such as aggregate DER response to a reserve activation from a transmission system operator). As long as the aggregate DER response remains within the specified range, we guarantee that there are no violations of the grid constraints, regardless of an aggregator chooses to disaggregate the power among DERs (i.e., it will not matter which individual DERs are asked to respond to the signal). Thus, the proposed method can be used to easily communicating limits on DER operations in the grid to aggregators and transmission system operators. This addresses important questions associated with the recently announced FERC Order 2222 [2], which mandates that DER aggregators should be allowed to participate in Independent System Operator (ISO) markets.

To illustrate the main idea behind the proposed method, we provide a simple motivating example. Consider the small power system illustrated in Fig. 1.1 where generators and a distribution grid are connected to the transmission system. The two generators G1 and G2 can bid into the electricity market and the amount of power provided by them combined is determined by solving a transmission system dispatch problem that considers the transmission constraints expressed us-



Figure 1.1: Example of transmission-distribution interaction.

ing  $P_{min}$ ,  $P_{max}$ . Analogous to this, the distribution grid comprises of two DER aggregators A1 and A2 that control DERs at multiple nodes as illustrated by the red and green circles, respectively. These aggregators can now bid into the electricity market. However, including the operational limits of each individual DER or individual DER aggregator in the transmission system dispatch problem can increase the computational complexity [3, 4, 5]. Thus, one promising alternative to model distribution grid capacity limits is for the DSO to identify the aggregate flexibility limits  $P_{min}$ ,  $P_{max}$  of power that can be provided from all DERs in the distribution grid while guaranteeing that no grid constraints are violated. More specifically, the DSO determines the ability of the distribution grid to modify the power exchanged with the transmission system at the point of interconnection (i.e. substation) while ensuring that the grid stays secure. These grid constraints can then be used analogous to transmission limits specified for individual lines and reduces the need to integrate a detailed distribution grid model.

#### 1.1 Literature Review: Existing Methods to Find Aggregate Power Flexibility

There are numerous works in literature which focus on identifying flexibility of individual DERs [6, 7] as well as system-level aggregate power flexibility [8, 9, 10] in distribution grids. Datadriven approaches employing Monte Carlo simulations with a large number of sampling scenarios are described in [11, 12, 13] to estimate the aggregate power flexibility range. Bilevel optimization models are used to improve the interactions between DSO and aggregators in [14, 15, 16, 17] as well as interactions between end consumers and aggregators in [17, 18]. Most of these methods assume that network constraints such as voltage magnitude limits are non-binding and hence, do not consider them. However, this might not be true for distribution grids with high penetrations of DERs where ensuring that voltage limits are not violated can be highly critical. The methods proposed in [19, 20] do consider network constraints to solve the flexibility aggregation and feasible disaggregation problem for a single time-step in the system. Geometric approaches that consider temporally coupled constraints are discussed in [21, 22, 23], where polytope sets are utilized to describe the power flexibility of an individual DERs. The aggregate power flexibility is then determined using polytopic projection in higher dimensional space or calculating the Minkowski sum of the polytope sets. It is important to note that the resulting mathematical problems are complex and cannot be solved efficiently. Furthermore, most of these previous works model the distribution grids using a balanced, single-phase equivalent power flow model. However, distribution grids are typically unbalanced due to untransposed lines, and asymmetrical three-phase loads [24]. To that end, [25, 26, 27, 28] employ the unbalanced three-phase network model to perform power flexibility aggregation for distribution grids.

#### **1.2 Main Contributions and Goals**

All the researches mentioned above assume that aggregators cooperate with the DSOs to maximize the DER flexibility. For any aggregate power flexibility determined by the DSO, if there exists a feasible disaggregation solution, it is assumed that the aggregators are aware of this solution and are required to adjust power injections of the DERs according to it. In practice, there may be multiple aggregators in the distribution network competing for the same capacity, and the DSO and aggregators are decision-makers who may make uncoordinated decisions according to partially conflicting,
individual objectives. In order to provide more freedom to the aggregators and transmission system operator to optimize their own outcomes (e.g. by choosing which DERs to control or deciding which aggregator to purchase flexibility from), the DSO can determine aggregate power flexibility such that *any* disaggregation solution (even the worst-case) will be feasible. This is the goal of the proposed method.

By definition, the aggregate flexibility will be smaller if we assume a worst-case disaggregation solution. As a result, the worst-case approach could lead to conservative solutions. However, it is possible to limit this conservativeness by determining simple rules for DER control activation, such as mandating that all DERs can only either increase or decrease their consumption or allowing the utility more control over DER reactive power output. These types of rules are easy to communicate to the aggregators. Furthermore, the aggregators have an incentive to follow them as it allows the DSO to increase the aggregate power flexibility range that can be offered to the grid, which works to the aggregator's benefit. Note that the DSO does not allocate capacity limits for individual DERs or individual DER aggregators, but instead determines total capacity limits for all DER devices combined. The individual DER aggregator capacity limits (and associated bids to provide flexibility) will be considered by the transmission system operator as part of the market clearing, which takes the distribution grid constraints as an input.

# 1.3 Overview of Bilevel Optimization Approach

Since this decision making process is hierarchial where an upper-level authority (i.e. DSO) takes decisions based on the unknown (and thus assumed to be worst-case) responses by the lower-level entity (i.e. aggregators), it is natural to leverage a bilevel optimization framework. We formulate a bilevel optimization problem which identifies the largest amount of active power flexibility that can be provided to the transmission system (i.e., at the substation). Our objective is to find the maximum allowable change in active power at the substation (relative to the current operating point) such that there are no voltage magnitude violations in the network, regardless of how this change in power injections is allocated by the aggregators among the individual DERs. This optimization problem is solved subject to several rules for DER active and reactive power control activation that can be easily communicated to aggregators. The upper-level problem maximizes the allowable power injections while guaranteeing that the voltage magnitudes stay within acceptable limits. The lowerlevel solves a three-phase OPF problem for each node in the network with objective to find the worst-case disaggregation solution that causes the largest (or smallest) voltage magnitude in the grid, subject to the power flow constraints and rules for DER activation. The lower-level problem is similar to the formulation proposed in [29], which we extend to include an unbalanced threephase power flow model and consideration of limits on DER control and the DER activation rules. We assume that the DSO enforces that all DERs are either increasing or decreasing their power output and also controls the reactive power setpoints of the DER inverters according to the IEEE Standard 1547-2018 [30]. We consider three different inverter reactive power control modes for reactive power control, namely constant power factor, constant reactive power and voltage-reactive power modes. We assume that the reactive power control setpoints are provided to the DERs either through the aggregator or directly from the DSO.

In the next few sections, we provide a brief overview of bilevel optimization and then present the formulation of the bilevel problem to identify aggregate power flexibility. We then describe the bilevel mathematical model and the analytical reformulation of the problem using linear approximations and strong duality. Finally, we test the proposed approach on two unbalanced distribution feeders.

### 1.4 Bilevel Optimization: Theoretical Background

A bilevel problem can be defined as an optimization problem that is constrained by another optimization problem. One of the main reasons to choose a bilevel optimization approach in our work is because we have two independent decision makers i.e. the DSO and DER aggregators and each of them has their own objective and constraints. By using this approach, we are able to capture the conflicting interests of the DSO and aggregators. To better explain how to solve such a problem with hierarchical structure, we consider a sequential game based on the Stackelberg game theory [31] where two players are involved. The first player is referred to as the leader problem with the decision vector represented by  $X_u$ . The decision vector of the second player or the follower problem is denoted by  $X_l$ . From a mathematical point of view, a general bilevel problem is given by

$$\max_{\boldsymbol{X}_{\mathbf{u}}} f_{\mathbf{u}}(\boldsymbol{X}_{\mathbf{u}}, \hat{\boldsymbol{X}}_{\mathbf{l}}) \tag{1.1a}$$

s.t. 
$$g_{\mathbf{u}}(\boldsymbol{X}_{\mathbf{u}}, \hat{\boldsymbol{X}}_{\mathbf{l}}) \le 0,$$
 (1.1b)

$$h_{\mathbf{u}}(\boldsymbol{X}_{\mathbf{u}}, \hat{\boldsymbol{X}}_{\mathbf{l}}) = 0, \qquad (1.1c)$$

where 
$$\hat{X}_{l} = \arg \max_{X_{l}} f_{l}(X_{u}, X_{l})$$
 (1.1d)

s.t. 
$$g_l(\boldsymbol{X}_u, \boldsymbol{X}_l) \le 0,$$
 (1.1e)

$$h_{\mathbf{l}}(\boldsymbol{X}_{\mathbf{u}}, \boldsymbol{X}_{\mathbf{l}}) = 0. \tag{1.1f}$$

Here, (1.1a)-(1.1c) describes the leader or the upper-level problem and (1.1d)-(1.1f) represents the follower or the the lower-level problem. It is important to note the optimal reaction of the follower is included in the decision making process of the leader in the bilevel problem. The leader problem has full knowledge of the objective and constraints of the follower problem, whereas the follower problem can only observe the decisions made by the leader problem and then optimize their own strategy. Therefore, the follower problem can be considered as an  $X_u$ -parameterized problem where the upper-level decision vector is assumed to be known and not considered as a variable. The solution pair ( $X_u$ ,  $\hat{X}_l$ ) where  $\hat{X}_l$  denotes the optimal response of the follower problem to the leader problem's decision  $X_u$  is feasible provided that  $\hat{X}_l$  satisfies the constraints (1.1b)-(1.1c) in the upper-level problem.

Although there is considerable body of literature on solving bilevel optimization problems, the hierarchical structure of the problem introduces challenges such as disconnectedness and non-convexity [32]. Even for simple examples of bilevel optimization tasks, it has been observed that the problems are not straightforward to handle mathematically. As a result, bilevel problems are considered to be strongly NP-hard to solve [33, 34]. Some of the classical approaches to solve

bilevel problems are summarized below:

- *Single-level reduction:* One of the most commonly used strategies is to reformulate the bilevel problem as a single-level problem. If the follower problem is convex, we can replace the lower-level optimization problem with its Karush-Kuhn-Tucker (KKT) conditions [35, 36, 37] or apply the strong duality theorem [38, 39, 40] for the lower-level problem. The advantage of this method is that the resulting single-level reformulation can be directly solved using state-of-the-art commercially available solvers.
- *Descent methods:* This is based on finding a descent direction in bilevel optimization that results in decrease in upper-level objective function value while ensuring that the solution is feasible [34, 41, 42]. A drawback of this approach is that finding the descent direction can be complex.
- *Penalty function methods:* This approach reformulates the problem as an unconstrained optimization task where a penalty term measures the extent of constraint violation [43, 44]. While the lower-level problem is replaced by a penalized problem, the bilevel structure is still maintained and as a result, the reduced problem is still challenging to solve.
- *Trust-region methods:* This is an iterative method where at every iteration, a model of the bilevel problem is built around the approximate solution [45, 46, 47]. If the approximation is accurate, the region is enlarged, otherwise it is reduced. Although this approach is popular due to the strong convergence properties, it can be computationally expensive since it involves solving the bilevel problem over multiple iterations.

Based on the advantages and limitations of the different approaches mentioned above, we choose to develop a linear bilevel problem which can be reformulated using the strong duality based, single-level reduction approach. We will discuss the reformulation in more detail in Section 2.3.

In this section, we assume that we have full information about the current operating point in the grid and the full system model (this assumption will be relaxed later in Section 3). Under this assumption, we describe our model of the flexibility offered by solar PV inverters and loads in the distribution grid and present a formulation of a bilevel optimization problem. The model we describe below is a linear model

# 2.1 System Modeling

We use phasor notation  $x = |x| \angle \theta = x_d + jx_q$ , where *x* is a complex phasor;  $|x|, \theta$  denote the magnitude and angle components;  $x_d, x_q$  represent the real and imaginary components, and  $j = \sqrt{-1}$ . The complex conjugate of *x* is represented by  $x^*$ . We consider a network with one slack node and  $\mathscr{N}$  denotes the set of remaining nodes with  $n = |\mathscr{N}|$ . In general, distribution grids comprise of several single- and two-phase nodes. Without loss of generality, we assume all nodes have three-phases with the set of phases defined by  $\Phi = \{a, b, c\}$ . For nodes with missing phases, the corresponding entries are set to zero. The total number of single-phase connections is represented by 3(n + 1). The distribution substation with index i = 0 is chosen as the slack node. Apart from the substation, the only other generators we consider in the network are single-phase, residential solar PV systems. To simplify notation, we assume that there is one solar PV inverter and one load at each single-phase, the corresponding entries are set to zero. All the optimization variables are denoted using bold letters and all vectors as well as matrices are represented using capital letters. We represent the element-wise product of two vectors using  $\odot$ . Given a vector  $X \in \mathbb{C}^n$ , we represent  $dg(X) \in \mathbb{C}^{(n \times n)}$  as a diagonal matrix with elements of *X* in its diagonal.

### 2.1.1 Current Operating Point

We consider the case where we have full information about the current operating point in the grid. At any three-phase node *i*, the voltage phasor is represented in rectangular form by  $V_i = V_{di} + jV_{qi}$ where  $V_{di} = \begin{bmatrix} v_{di}^a & v_{di}^b & v_{di}^c \end{bmatrix}^\top$  denotes the real component and  $V_{qi} = \begin{bmatrix} v_{qi}^a & v_{qi}^b & v_{qi}^c \end{bmatrix}^\top$  is the imaginary component. Similarly, we can define the three-phase active and reactive power injections by  $P_i = \begin{bmatrix} p_i^a & p_i^b & p_i^c \end{bmatrix}^\top$  and  $Q_i = \begin{bmatrix} q_i^a & q_i^b & q_i^c \end{bmatrix}^\top$ , respectively. At any node *i*, we denote the current operating point using  $V_{di}, V_{qi}, P_i, Q_i \in \mathbb{R}^3$ .

#### 2.1.2 Power Injections

The power injections at node  $i \in \mathcal{N}$  connected to phase  $\phi \in \Phi$  are expressed as the difference between the generation and load demand using

$$p_{i}^{\phi} = p_{\mathrm{G},i}^{\phi} - p_{\mathrm{L},i}^{\phi}, \qquad (2.1a)$$

$$q_i^{\phi} = q_{\mathrm{G},i}^{\phi} - q_{\mathrm{L},i}^{\phi}, \qquad (2.1b)$$

where  $p_{G,i}^{\phi}$ ,  $q_{G,i}^{\phi}$  are the active and reactive power generation by solar PV inverters, respectively;  $p_{L,i}^{\phi}$ ,  $q_{L,i}^{\phi}$  denote the respective active and reactive load demand.

The power balance is maintained by the substation, which supplies the difference between the load and power generation as well as other losses in the grid. At the current operating point, the active and reactive power injection at the substation are denoted by  $P_{G,0}$ ,  $Q_{G,0} \in \mathbb{R}^3$ , respectively.

### 2.1.3 Voltage Representation

The voltage variables are expressed in rectangular form. For any node  $i \in \mathcal{N}$ , the real and imaginary components of voltage are given by  $V_{di}$ ,  $V_{qi} \in \mathbb{R}^3$ , respectively. The distribution substation is assumed to be a balanced three-phase node where the voltage is fixed at

$$\boldsymbol{V}_{\mathbf{d}0} + j \cdot \boldsymbol{V}_{\mathbf{q}0} = \begin{bmatrix} e^{j0^{\circ}} & e^{-j120^{\circ}} & e^{j120^{\circ}} \end{bmatrix}^{\top}.$$
 (2.2)

For every other node  $i \in \mathcal{N}$  connected to phase  $\phi$ , we include additional voltage magnitude variables  $|v_i^{\phi}|$  in the formulation. These voltage magnitude variables are quadratic functions of the rectangular voltage variables  $v_{di}^{\phi}, v_{qi}^{\phi}$ . Note that we want to formulate a linear bilevel problem so that we can reformulate it using the strong duality based approach. Hence, we need to linearize the quadratic constraints. Given the current voltage magnitude  $|v_i^{\phi}| = \sqrt{(v_{di}^{\phi})^2 + (v_{qi}^{\phi})^2}$ , we use First-order Taylor approximation to get the following linear constraints for the voltage magnitude:

$$(v_{di}^{\phi})^{2} + (v_{qi}^{\phi})^{2} + 2v_{di}^{\phi}v_{di}^{\phi} + 2v_{qi}^{\phi}v_{qi}^{\phi} = |v_{i}^{\phi}|^{2} + 2|v_{i}^{\phi}||v_{i}^{\phi}|.$$
(2.3)

#### 2.1.4 Power Flow

Following [48], the critical components of distribution grid such as distribution lines, cables, and transformers etc. are modeled using a nodal admittance matrix,

$$Y = G + jB = \begin{bmatrix} Y_{00} & Y_{0L} \\ Y_{L0} & Y_{LL} \end{bmatrix},$$
(2.4)

where  $G, B \in \mathbb{R}^{3(n+1)\times 3(n+1)}$  represent the nodal conductance and susceptance matrices, respectively.  $Y_{00} \in \mathbb{C}^{3\times 3}$  denotes the submatrix corresponding to the slack node and  $Y_{LL} \in \mathbb{C}^{3n\times 3n}$  represents the submatrix of the entire network ignoring the slack node.  $Y_{0L} \in \mathbb{C}^{3\times 3n}, Y_{L0} \in \mathbb{C}^{3n\times 3}$  are the other submatrices of the nodal admittance matrix *Y*.

The three-phase power flow equations at any node *i* can be expressed as

$$\boldsymbol{P}_{i} = \sum_{k \in \{0,\mathcal{N}\}} \boldsymbol{V}_{\mathbf{d}i} \odot \left[ \boldsymbol{G}_{ik} \boldsymbol{V}_{\mathbf{d}k} - \boldsymbol{B}_{ik} \boldsymbol{V}_{\mathbf{q}k} \right] + \sum_{k \in \{0,\mathcal{N}\}} \boldsymbol{V}_{\mathbf{q}i} \odot \left[ \boldsymbol{B}_{ik} \boldsymbol{V}_{\mathbf{d}k} + \boldsymbol{G}_{ik} \boldsymbol{V}_{\mathbf{q}k} \right],$$
(2.5a)

$$\boldsymbol{Q}_{i} = \sum_{k \in \{0,\mathcal{N}\}} \boldsymbol{V}_{\mathbf{d}i} \odot \left[ -B_{ik} \boldsymbol{V}_{\mathbf{d}k} - G_{ik} \boldsymbol{V}_{\mathbf{q}k} \right] + \sum_{k \in \{0,\mathcal{N}\}} \boldsymbol{V}_{\mathbf{q}i} \odot \left[ G_{ik} \boldsymbol{V}_{\mathbf{d}k} - B_{ik} \boldsymbol{V}_{\mathbf{q}k} \right],$$
(2.5b)

where  $G_{ik}, B_{ik} \in \mathbb{R}^{3 \times 3}$  are the conductance and susceptance submatrices calculated using (2.4).

Due to the non-linearity and non-convexity of the power flow equations described above, we leverage a linear approximation to be used instead of (2.5) in our linear bilevel problem. To linearize the power flow equations, we employ the fixed-point (FP) linearization described in [49]. We choose this linear power flow model because it exhibits better global approximation accuracy when compared to commonly used linear models based on first-order Taylor series expansion [49]. In order to express the fixed-point power flow equations, we first separate the complex power injections at all nodes except the slack node into wye-connected injections  $S_Y \in \mathbb{C}^{3n}$  and delta-connected injections  $S_\Delta \in \mathbb{C}^{3n}$ . The fixed-point equation can be expressed as a function of the current operating voltage phasor  $V \in \mathbb{C}^{3n}$  and the complex power injections to get

$$V_{\mathbf{d}} + jV_{\mathbf{q}} = f(V, S_{\mathbf{Y}}, S_{\Delta}) = W + Y_{\mathrm{LL}}^{-1} \Big( dg(V^*)^{-1} S_{\mathbf{Y}}^* + H^{\top} dg(HV^*)^{-1} S_{\Delta}^* \Big),$$
(2.6)

where  $V_d, V_q \in \mathbb{C}^{3n}$  represent the vectors of voltage variables at all nodes except the substation;  $W = -Y_{LL}^{-1}Y_{L0}V_0$  is the no-load voltage phasor and  $H \in \mathbb{Z}^{3n \times 3n}$  is a transformation matrix as defined in [49]. The linear fixed-point power flow equations in (2.6) can be rewritten by rearranging and equating the real and imaginary terms to get

$$\boldsymbol{V}_{\mathbf{d}} = \Re \left\{ Z_1 \right\} + \Re \left\{ Z_2 \right\} \cdot \boldsymbol{P}_{\mathbf{Y}} + \Im \left\{ Z_2 \right\} \cdot \boldsymbol{Q}_{\mathbf{Y}}, \tag{2.7a}$$

$$\boldsymbol{V}_{\boldsymbol{q}} = \Im\left\{Z_{1}\right\} + \Im\left\{Z_{2}\right\} \cdot \boldsymbol{P}_{\boldsymbol{Y}} - \Re\left\{Z_{2}\right\} \cdot \boldsymbol{Q}_{\boldsymbol{Y}}, \qquad (2.7b)$$

where 
$$Z_1 = -W + Y_{\text{LL}}^{-1} H^{\top} dg (HV^*)^{-1} S_{\Delta}^*$$
,  $Z_2 = Y_{\text{LL}}^{-1} dg (V^*)^{-1}$ .

Here,  $Z_1, Z_2$  are fixed parameters, while  $P_Y$  is the real component of  $S_Y$  and includes the active power flexibility from loads and PV inverters. The imaginary component of  $S_Y$ , which is denoted by  $Q_Y$ , includes the reactive power injections of PV inverters and load reactive power demand.

### 2.1.5 Flexibility Modelling

The aggregators can offer flexibility by modifying the active power generation of the solar PV inverters and active power demand of controllable loads.

#### **Modelling of Loads**

For a load at node  $i \in \mathcal{N}$  connected to phase  $\phi \in \Phi$ , let  $\Delta p_{\mathbf{L},i}^{\phi}$  denote the change in load from the current operating point such that

$$\underline{p}_{\mathbf{L},i}^{\phi} \le p_{\mathbf{L},i}^{\phi} + \Delta p_{\mathbf{L},i}^{\phi} \le \overline{p}_{\mathbf{L},i}^{\phi}.$$
(2.8)

Here,  $\underline{p}_{\mathrm{L},i}^{\phi}, \overline{p}_{\mathrm{L},i}^{\phi}$  represent the minimum and maximum limits for the active power load demand. We assume that the loads operate with a constant power factor  $pf_{\mathrm{L},i}^{\phi}$ , which means the reactive power consumption is given by

$$\boldsymbol{q}_{\mathbf{L},i}^{\phi} = \frac{\sqrt{1 - (pf_{\mathbf{L},i}^{\phi})^{2}}}{pf_{\mathbf{L},i}^{\phi}} \cdot \left(p_{\mathbf{L},i}^{\phi} + \Delta \boldsymbol{p}_{\mathbf{L},i}^{\phi}\right).$$
(2.9)

#### **Modeling of Solar PV Active Power Generation**

Similar to the load modelling, we consider  $\Delta p_{\mathbf{G},i}^{\phi}$  as the change in active power generation from the current operating point for a solar PV inverter at node *i* connected to phase  $\phi$  such that

$$\underline{p}_{\mathbf{G},i}^{\phi} \le p_{\mathbf{G},i}^{\phi} + \Delta p_{\mathbf{G},i}^{\phi} \le \overline{p}_{\mathbf{G},i}^{\phi}, \tag{2.10}$$

where  $\underline{p}_{G,i}^{\phi}, \overline{p}_{G,i}^{\phi}$  denote the respective minimum and maximum limits for the active power generation. Reactive power injections from the solar PV inverters are further discussed below.

#### **Aggregate Power Flexibility**

The total active power flexibility  $\Delta p$  provided by the distribution grid can be bounded to define the secure operating range using

$$\Delta \boldsymbol{p}^{-} \leq \underbrace{\sum_{i \in \mathcal{N}} \sum_{\phi \in \Phi} \left( \Delta \boldsymbol{p}_{\mathbf{G},i}^{\phi} - \Delta \boldsymbol{p}_{\mathbf{L},i}^{\phi} \right)}_{\Delta \boldsymbol{p}} \leq \Delta \boldsymbol{p}^{+}, \tag{2.11}$$

where  $\Delta p^-$ ,  $\Delta p^+$  are the respective lower and upper limits of the aggregate power flexibility which will be provided by the substation since there are no other generators in the network apart from the solar PV systems. Note that the current operating point can be considered as the condition when the substation offers zero flexibility to the transmission network (i.e.  $\Delta p = 0$ ).

To find the limits on the aggregate power flexibility, we enforce a rule that the active power injection of each solar PV inverter as well as each load controlled by the aggregator is adjusted in the same way. The reasoning behind this rule is to ensure that limiting the aggregate response is sufficient to actually control voltage magnitude violations in the feeder. Without this rule, some DERs in one part of the feeder can arbitrarily increase their injections as long as the increases are offset by an equally large decrease in injections elsewhere. In this case, even a zero change in the overall active power injection can cause voltage violations.

To model this rule in our optimization problem, we consider two cases:

(a) *Positive* (+) *case:* This case corresponds to finding the maximum aggregate power flexibility limit  $\Delta p^+ \ge 0$ . The active power injections at any node  $i \in \mathcal{N}$  connected to phase  $\phi$  is required

to be greater than or equal to the current value  $p_i^{\phi}$ . This means that the load demand is reduced and that any solar PV inverters that were not operating at full capacity at the current operating point can increase their active power injection. This gives rise to the following constraints on flexibility activation for PV generation and load:

$$\Delta \boldsymbol{p}_{\mathbf{L},i}^{\phi} \leq 0, \ \Delta \boldsymbol{p}_{\mathbf{G},i}^{\phi} \geq 0.$$
(2.12)

(b) Negative (-) case: This case is used to calculate the minimum aggregate power flexibility limit Δp<sup>-</sup> ≤ 0. We assume that the active power injections at every node i ∈ N connected to phase φ is lower than the current value p<sup>φ</sup><sub>i</sub>. This case gives rise to the following rules for flexibility activation from PV generation and load:

$$\Delta \boldsymbol{p}_{\mathbf{L},i}^{\phi} \ge 0, \ \Delta \boldsymbol{p}_{\mathbf{G},i}^{\phi} \le 0.$$
(2.13)

# 2.1.6 Reactive Power Control from Solar PV Inverters

We assume that all solar PV inverters are smart inverters with the ability to provide reactive power control as specified in the IEEE Standard 1547-2018 [30]. To determine the reactive power injections, which are decision variables in our optimization problem, we consider three modes in which the PV inverters can operate. For all three modes, any inverter at node *i* connected to phase  $\phi$  should not exceed the the inverter apparent power capacity  $|s_{G,i}^{\phi}|$ . Since the inverter limits are quadratic constraints, we use the circular constraint linearization method to outer approximate the feasible region using linear constraints given by

$$0 \le p_{\mathbf{G},i}^{\phi} + \Delta \boldsymbol{p}_{\mathbf{G},i}^{\phi} \le |s_{\mathbf{G},i}^{\phi}|, \qquad (2.14a)$$

$$-|s_{\mathbf{G},i}^{\phi}| \le q_{\mathbf{G},i}^{\phi} \le |s_{\mathbf{G},i}^{\phi}|, \tag{2.14b}$$

$$p_{\mathbf{G},i}^{\phi} + \Delta \boldsymbol{p}_{\mathbf{G},i}^{\phi} + \boldsymbol{q}_{\mathbf{G},i}^{\phi} \le \sqrt{2} \cdot |s_{\mathbf{G},i}^{\phi}|, \qquad (2.14c)$$

$$p_{G,i}^{\phi} + \Delta p_{G,i}^{\phi} - q_{G,i}^{\phi} \le \sqrt{2} \cdot |s_{G,i}^{\phi}|.$$
(2.14d)

Note that it is possible to obtain a more accurate representation of the quadratic inverter constraints by including more linear constraints.

We will next describe the three inverter reactive power control modes. For each mode, the setpoints provided by the DSO are different and will be considered as the leader problem variables. All other variables in the inverter constraints are part of the follower problem. The assignment of the inverter variables to the upper-level and lower-level problems for different inverter reactive power control modes is summarized in Table 2.1 at the end of this subsection.

# **Constant Power Factor Mode**

When the inverter is operating in this mode, the solar PV inverters operate at a constant power factor. The power factor setting can be adjusted locally or remotely as specified by the DSO. In

this work, we assume that DSO provides the target power factor setting to the DERs. The reactive power injection  $q_{G,i}^{\phi}$  depends on the active power generation and the target power factor  $pf_{G,i}^{\phi}$  and expressed as

$$\boldsymbol{q}_{\mathbf{G},i}^{\phi} = \boldsymbol{\gamma}_{\mathbf{G},i}^{\phi} \cdot \left( \boldsymbol{p}_{\mathbf{G},i}^{\phi} + \Delta \boldsymbol{p}_{\mathbf{G},i}^{\phi} \right), \qquad (2.15a)$$

$$\frac{-\sqrt{1 - (pf_{G,i}^{\phi})^{2}}}{pf_{G,i}^{\phi}} \le \gamma_{G,i}^{\phi} \le \frac{\sqrt{1 - (pf_{G,i}^{\phi})^{2}}}{pf_{G,i}^{\phi}},$$
(2.15b)

where  $\gamma^{\phi}_{\mathbf{G},i}$  is an upper-level variable which defined as the power ratio of the inverter.

Note that (2.15a) consist of bilinear terms involving the upper-level variables  $\gamma_{\mathbf{G},i}^{\phi}$  and lower-level variables  $\Delta p_{\mathbf{G},i}^{\phi}$ . Since the upper-level variables are considered as parameters in the follower problem, (2.15) is linear for the lower-level problem.

### **Constant Reactive Power Mode**

In this mode, the reactive power injection  $q_{G,i}^{\phi}$  at node *i* connected to phase  $\phi$  is specified by the DSO and hence, considered as an upper-level variable. The DERs are allowed to operate at any power factor in this mode as long as the power factor requirements are met at the point of interconnection to the grid. In our work, we assume that PV inverters are operating within the specified power ratio  $\gamma_{G,i}^{\phi}$  and we enforce limits on the reactive power using

$$-\gamma_{\mathbf{G},i}^{\phi} \cdot \left( p_{\mathbf{G},i}^{\phi} + \Delta \boldsymbol{p}_{\mathbf{G},i}^{\phi} \right) \leq \boldsymbol{q}_{\mathbf{G},i}^{\phi} \leq \gamma_{\mathbf{G},i}^{\phi} \cdot \left( p_{\mathbf{G},i}^{\phi} + \Delta \boldsymbol{p}_{\mathbf{G},i}^{\phi} \right).$$
(2.16)

#### **Voltage-Reactive Power Mode**

This mode is also commonly referred to as voltage regulation with reactive droop. The reactive power injection of the solar PV inverter is a function of the voltage magnitude, which follows a piecewise linear characteristic with the parameter values specified by the DSO. An example voltage magnitude-reactive power characteristic is illustrated in Fig. 2.1.

Consider a PV inverter at node *i* connected to phase  $\phi$  with the maximum available reactive power  $\overline{q}_{\mathbf{G},i}^{\phi}$  specified by the DSO and included in our bilevel task as an upper-level decision variable. If the actual voltage magnitude  $|v_i^{\phi}|$  at this single-phase connection is below the minimum limit  $\underline{v}$ , the PV inverter injects all the available reactive power  $\overline{q}_{\mathbf{G},i}^{\phi}$ . Conversely, the PV inverter absorbs the maximum reactive power when the voltage magnitude is above the maximum limit  $\overline{v}$ . In the grey shaded region, the reactive power can be expressed as a function of the voltage magnitude using

$$\boldsymbol{q}_{\mathbf{G},i}^{\phi} = \overline{\boldsymbol{q}}_{\mathbf{G},i}^{\phi} - 2\overline{\boldsymbol{q}}_{\mathbf{G},i}^{\phi} \cdot \left(\frac{|\boldsymbol{v}_{i}^{\phi}| - \underline{v}}{\overline{v} - \underline{v}}\right), \qquad (2.17a)$$

$$0 \le \overline{\boldsymbol{q}}_{\mathbf{G},i}^{\phi} \le |\boldsymbol{s}_{\mathbf{G},i}^{\phi}|. \tag{2.17b}$$

It is quite challenging to model the linear piecewise equality constraints illustrated in Fig. 2.1 in an optimization problem. In our work, we assume that the inverter is operating only in the gray shaded region and hence, we can use (2.17) to determine the reactive power injections. This is a reasonable assumption since the voltage magnitude  $|v_i^{\phi}|$  is supposed to be within the limits. It is important to note here that the actual upper bound of the maximum available reactive power  $\overline{q}_{G,i}^{\phi}$  can be expressed as  $|s_{G,i}^{\phi}|^2 - (p_{G,i}^{\phi} + \Delta p_{G,i}^{\phi})^2$ . Since this expression is quadratic, we relax the upper bound and come up with a linear constraint as defined in (2.17b).



Figure 2.1: Voltage magnitude-reactive power characteristic

Similar to the constant power factor mode, the reactive power constraints in (2.17a) consist of bilinear terms involving the upper-level variables  $\bar{q}_{G,i}^{\phi}$  and lower-level variables  $|v_i^{\phi}|$ .

Reactive power mode	Upper-level $(X_u)$	Lower-level $(X_l)$
Constant power factor	$\gamma^{\phi}_{{f G},i}$	$\Delta oldsymbol{p}^{\phi}_{\mathbf{G},i},oldsymbol{q}^{\phi}_{\mathbf{G},i}$
Constant reactive power	$\boldsymbol{q}^{\phi}_{\mathbf{G},i}$	$\Delta oldsymbol{p}^{\phi}_{\mathbf{G},i}$
Voltage-reactive power	$\overline{oldsymbol{q}}^{\phi}_{\mathbf{G},i}$	$\Delta oldsymbol{p}^{\phi}_{\mathbf{G},i},oldsymbol{q}^{\phi}_{\mathbf{G},i},oldsymbol{v}^{\phi}_iert$

Table 2.1: Assignment of inverter variables to the bilevel problem for different inverter reactive power control modes.

#### 2.2 Bilevel Optimization Problem

The main objective of the bilevel problem is to find the maximum range of aggregate power flexibility while ensuring that the grid is secure even in the worst-case conditions. In this section, we first discuss the bilevel problem formulation where inverters are operating in constant power factor mode and DSO controls the power ratio  $\gamma_{\mathbf{G},i}^{\phi}$  of each inverter. The upper-level variable vector  $\mathbf{X}_{\mathbf{u}}$ and lower-level variable vector  $\mathbf{X}_{\mathbf{l}}$  of the bilevel problem is denoted by

$$\boldsymbol{X}_{\mathbf{u}} := \{\Delta \boldsymbol{p}^{+}, \Delta \boldsymbol{p}^{-}, \boldsymbol{\gamma}_{\mathbf{G},i}^{\phi} \forall_{\phi \in \Phi, i \in \mathcal{N}}\}, \\
\boldsymbol{X}_{\mathbf{l}} := \left\{\{\boldsymbol{v}_{\mathbf{d}i}^{\phi}, \boldsymbol{v}_{\mathbf{q}i}^{\phi}, |\boldsymbol{v}_{i}^{\phi}|, \Delta \boldsymbol{p}_{\mathbf{G},i}^{\phi}, \Delta \boldsymbol{p}_{\mathbf{L},i}^{\phi}, \boldsymbol{q}_{\mathbf{G},i}^{\phi}\}_{\forall_{\phi \in \Phi, i \in \mathcal{N}}}\right\}.$$
(2.18)

Recall that we define a positive and negative case in Section 2.1.5 to enforce rules on how all aggregators are required to control the DER power injections in the same way. The aggregate power flexibility and inverter setpoints determined by the bilevel problem must be applicable for both the cases and hence, we only need one set of the upper-level variables  $X_u$  that is shared by both cases. On the other hand, the worst-case conditions determined for the positive case can be different from the worst-case conditions seen for the negative case because the rules for power injections are not same. Furthermore, we determine the worst-case conditions at every single-phase connection in the network by finding the minimum and maximum voltage magnitude achievable at that connection. Again, these operating conditions will be different for the positive and negative cases. As a result, we need to consider four sets of lower-level variables  $X_1^+, X_1^-, \overline{X}_1^+, \overline{X}_1^-$  which represent four different scenarios as summarized in Table 2.2. For example, scenario 1 corresponds to the positive case where the worst-case condition is determined by finding the minimum achievable voltage magnitude.

Scenario (#)	Case	Voltage	Lower-level
1	Dogitivo	Min.	$\underline{X}_{\mathbf{l}}^+$
2	FOSITIVE	Max.	$\overline{X}_{\mathbf{l}}^+$
3	Negative	Min.	$\underline{X}_{\mathbf{l}}^{-}$
4	negative	Max.	$\overline{X}_{l}^{-}$

Table 2.2: Summary of lower-level problem variables corresponding to each single-phase connection in the network.

We define four sets of lower-level system constraints which depend on the follower problem variable vectors  $\underline{X}_{l}^{+}, \underline{X}_{l}^{-}, \overline{X}_{l}^{+}, \overline{X}_{l}^{-}$  as

$$\mathbb{L}(\underline{X}_{l}^{+}), \mathbb{L}(\underline{X}_{l}^{-}), \mathbb{L}(\overline{X}_{l}^{+}), \mathbb{L}(\overline{X}_{l}^{-}) := \begin{cases} \text{Voltage constraints (2.3),} \\ \text{Load reactive power constraints (2.9),} \\ \text{Power flow (2.7).} \end{cases}$$

The inverter constraints are expressed using both the leader and follower problem variables. For the inverter at node *i* connected to phase  $\phi$  operating in the constant power factor mode, we can represent four sets of inverter constraints using

$$\mathbb{I}(\boldsymbol{\gamma}_{\mathbf{G},i}^{\phi}, \underline{\mathbf{X}}_{\mathbf{l}}^{+}), \ \mathbb{I}(\boldsymbol{\gamma}_{\mathbf{G},i}^{\phi}, \underline{\mathbf{X}}_{\mathbf{l}}^{-}), \ \mathbb{I}(\boldsymbol{\gamma}_{\mathbf{G},i}^{\phi}, \overline{\mathbf{X}}_{\mathbf{l}}^{+}), \ \mathbb{I}(\boldsymbol{\gamma}_{\mathbf{G},i}^{\phi}, \overline{\mathbf{X}}_{\mathbf{l}}^{-}) := \text{Inverter constraints (2.14), (2.15).}$$

Similarly, the active power flexibility limits are defined using both the leader and follower problem variables. We denote the sets of active power flexibility constraints for the positive and negative case using

$$\mathbb{F}(\Delta \boldsymbol{p}^{+}, \Delta \boldsymbol{p}^{-}, \underline{\boldsymbol{X}}_{\mathbf{l}}^{+}), \ \mathbb{F}(\Delta \boldsymbol{p}^{+}, \Delta \boldsymbol{p}^{-}, \overline{\boldsymbol{X}}_{\mathbf{l}}^{+}) := \text{Positive case (2.11), (2.12),}$$
$$\mathbb{F}(\Delta \boldsymbol{p}^{+}, \Delta \boldsymbol{p}^{-}, \underline{\boldsymbol{X}}_{\mathbf{l}}^{-}), \ \mathbb{F}(\Delta \boldsymbol{p}^{+}, \Delta \boldsymbol{p}^{-}, \overline{\boldsymbol{X}}_{\mathbf{l}}^{-}) := \text{Negative case (2.11), (2.13).}$$

The bilevel problem to determine the aggregate power flexibility limits  $\Delta p^+, \Delta p^-$  is formulated as

$$\begin{split} \max_{\mathbf{X}_{u}} \Delta p^{+} - \Delta p^{-} \quad & (\mathbf{P}^{\pm}) \\ \text{s.t. } \underline{v} \leq |v_{i,\min}^{\phi}|^{+}, |v_{i,\max}^{\phi}|^{+} \leq \overline{v}, \quad \forall_{\phi \in \Phi, i \in \mathscr{N}}, \\ \underline{v} \leq |v_{i,\min}^{\phi}|^{-}, |v_{i,\max}^{\phi}|^{-} \leq \overline{v}, \quad \forall_{\phi \in \Phi, i \in \mathscr{N}}, \\ \text{where } |v_{i,\min}^{\phi}|^{+} = \min_{\mathbf{X}_{1}^{+}} |v_{i}^{\phi}|^{+}, \quad \forall_{\phi \in \Phi, i \in \mathscr{N}}, \\ \text{s.t. System constraints } \mathbb{L}(\mathbf{X}_{1}^{+}), \\ \text{Inverter constraints } \mathbb{F}(\Delta p^{+}, \Delta p^{-}, \mathbf{X}_{1}^{+}), \\ |v_{i,\max}^{\phi}|^{+} = \max_{\mathbf{X}_{1}^{+}} |\overline{v}_{i}^{\phi}|^{+}, \quad \forall_{\phi \in \Phi, i \in \mathscr{N}}, \\ \text{s.t. System constraints } \mathbb{L}(\mathbf{X}_{1}^{+}), \\ \text{Inverter constraints } \mathbb{I}(\gamma_{G,i}^{\phi}, \mathbf{X}_{1}^{+}), \\ \text{Inverter constraints } \mathbb{I}(\mathbf{X}_{1}^{-}), \\ \text{Inverter constraints } \mathbb{I}(\mathbf{X}_{0}^{-}, \Delta p^{-}, \mathbf{X}_{1}^{+}), \\ |\underline{v}_{i,\min}^{\phi}|^{-} = \min_{\mathbf{X}_{1}^{-}} |\underline{v}_{i}^{\phi}|^{-}, \quad \forall_{\phi \in \Phi, i \in \mathscr{N}}, \\ \text{s.t. System constraints } \mathbb{L}(\mathbf{X}_{1}^{-}), \\ \text{Inverter constraints } \mathbb{I}(\Delta p^{+}, \Delta p^{-}, \mathbf{X}_{1}^{-}), \\ |\overline{v}_{i,\max}^{\phi}|^{-} = \max_{\mathbf{X}_{1}^{-}} |\overline{v}_{i}^{\phi}|^{-}, \quad \forall_{\phi \in \Phi, i \in \mathscr{N}}, \\ \text{s.t. System constraints } \mathbb{I}(\Delta p^{+}, \Delta p^{-}, \mathbf{X}_{1}^{-}), \\ |\overline{v}_{i,\max}^{\phi}|^{-} = \max_{\mathbf{X}_{1}^{-}} |\overline{v}_{i}^{\phi}|^{-}, \quad \forall_{\phi \in \Phi, i \in \mathscr{N}}, \\ \text{s.t. System constraints } \mathbb{I}(\Delta p^{+}, \Delta p^{-}, \mathbf{X}_{1}^{-}), \\ |\overline{v}_{i,\max}^{\phi}|^{-} = \max_{\mathbf{X}_{1}^{-}} |\overline{v}_{i}^{\phi}|^{-}, \quad \forall_{\phi \in \Phi, i \in \mathscr{N}}, \\ \text{s.t. System constraints } \mathbb{I}(\mathbf{X}_{1}^{-}), \\ \text{Inverter constraints } \mathbb{I}(\mathbf{X}_{1}^{-}), \\ \text{Inverter constraints } \mathbb{I}(\nabla p_{i}^{-}, \mathbf{X}_{1}^{-}), \\ \text{Inverter constraints } \mathbb{I}(\nabla p_{i}^{-}, \mathbf{X}_{1}^{-}), \\ \text{Flexibility constraints } \mathbb{I}(\Delta p^{+}, \Delta p^{-}, \mathbf{X}_{1}^{-}). \\ \end{array}$$

Note that for each node  $i \in \mathcal{N}$  connected to phase  $\phi \in \Phi$ , we need to include four lower-level problems corresponding to the four scenarios described in Table 2.2. Hence, the total number of lower-level problems in  $(P^{\pm})$  is 12n.

### 2.3 Single-level Reformulation

One of the most frequently used solution method for bilevel optimization is to reformulate the problem into a single-level problem. In this section, we utilize the strong-duality theorem for the lower-level problem to perform the single-level reduction. Given that the lower-level problem has a finite optimal solution, the dual feasible set is non empty and strong duality holds for every primal and dual feasible pairs. This strong duality based reformulation does not require the use of any other additional properties or assumptions. Hence, it can be less restrictive than the KKT conditions based reformulation [50].

For simplicity of exposition, we first derive the reformulation for a setting with only one follower problem and reactive power control using a constant power factor. Then, we discuss how the reformulation generalizes to the setting with multiple follower problems and other reactive power control modes.

### 2.3.1 Reformulation with Single Follower Problem

Consider a reduced version of the bilevel problem  $P^{\pm}$  where the objective of the upper-level problem is to determine the aggregate power flexibility limit  $\Delta p^+$  for the positive case only. We only consider one follower problem with the objective to find the maximum achievable voltage magnitude at node *i* connected to phase  $\phi$ . The simplified bilevel problem can be represented as

$$\begin{split} \max_{X_{u}} \Delta p^{+} & (P^{+}) \\ \text{s.t. } |v_{i,\max}^{\phi}|^{+} \leq \overline{v}, \\ \text{where } |v_{i,\max}^{\phi}|^{+} = \max_{\overline{X}_{1}^{+}} |\overline{v}_{i}^{\phi}|^{+} \\ \text{s.t. System constraints } \mathbb{L}(\overline{X}_{1}^{+}), \\ \text{Inverter constraints } \mathbb{I}(\gamma_{\mathbf{G},i}^{\phi}, \overline{X}_{1}^{+}), \\ \text{Flexibility constraints } \mathbb{F}(\Delta p^{+}, \overline{X}_{1}^{+}). \end{split}$$

To derive the strong duality based reformulation of the reduced problem  $P^+$ , we start by rewriting the bilevel problem  $P^+$  in a more abstract form to get

$$\max_{\Delta \boldsymbol{p}, \boldsymbol{\gamma}} \Delta \boldsymbol{p} \tag{2.19a}$$

s.t. 
$$\hat{\boldsymbol{x}}_{\mathbf{l}} \le \boldsymbol{b}_1,$$
 (2.19b)

where 
$$\hat{x}_{l} = \max_{x_{l}} x_{l}$$
 (2.19c)

s.t. 
$$a_2 \boldsymbol{x_l} \le b_2$$
, (2.19d)

$$a_3 \boldsymbol{x_l} + d_3 \boldsymbol{\gamma} \boldsymbol{x_l} \le b_3, \qquad (2.19e)$$

$$a_4 \boldsymbol{x}_{\mathbf{l}} + d_4 \Delta \boldsymbol{p} \le b_4, \tag{2.19f}$$

where  $x_{l}$  is the follower problem variable and  $\gamma, \Delta p$  represent the upper-level variables which are considered as parameters in the lower-level problem. Therefore, the follower problem is linear in  $x_{l}$  even though we have bilinear terms in (2.19e). By taking dual of the lower-level problem (2.19c)-(2.19f), we get

$$\min_{\boldsymbol{\lambda}} \begin{bmatrix} b_2 & b_3 & (b_4 - d_4 \Delta \boldsymbol{p}) \end{bmatrix} \boldsymbol{\lambda}$$
(2.20a)

s.t. 
$$\begin{bmatrix} a_2 & (a_3 + d_3 \boldsymbol{\gamma}) & a_4 \end{bmatrix} \boldsymbol{\lambda} = 1,$$
 (2.20b)

$$\boldsymbol{\lambda} \ge 0, \tag{2.20c}$$

where  $\lambda$  is the dual variable vector. For a given decision  $\Delta p$  of the leader problem, weak duality of the follower problem states that the dual objective provides an upper bound for the primal objective, i.e.

$$x_{\mathbf{l}} \leq \begin{bmatrix} b_2 & b_3 & (b_4 - d_4 \Delta p) \end{bmatrix} \lambda$$
 (2.21)

must hold for any feasible primal-dual pair  $(x_1, \lambda)$ . Using the strong duality theorem, we know that the feasible primal-dual pair is the optimal solution when the objective value of the primal problem  $\hat{x}_1$  is equal to the objective value of the dual problem. This occurs when

$$\boldsymbol{x}_{\mathbf{l}} \geq \begin{bmatrix} b_2 & b_3 & (b_4 - d_4 \Delta \boldsymbol{p}) \end{bmatrix} \boldsymbol{\lambda}.$$
 (2.22)

Consequently, we can now replace the dual of the lower-level problem (2.20) by the following set of constraints:

$$\boldsymbol{x}_{\mathbf{l}} \geq \begin{bmatrix} b_2 & b_3 & (b_4 - d_4 \Delta \boldsymbol{p}) \end{bmatrix} \boldsymbol{\lambda},$$
 (2.23a)

$$\begin{bmatrix} a_2 & (a_3+d_3\gamma) & a_4 \end{bmatrix} \boldsymbol{\lambda} = 1, \qquad (2.23b)$$

$$\boldsymbol{\lambda} \ge 0, \tag{2.23c}$$

We can use the above set of constraints and reformulate the bilevel problem (2.19) to get a single-

level optimization problem given by

$$\max_{\Delta p, \gamma, x_1, \lambda} \Delta p \tag{2.24a}$$

s.t. 
$$x_{\mathbf{l}} \le b_1$$
, (2.24b)

$$a_2 \boldsymbol{x_l} \le b_2, \tag{2.24c}$$

$$a_3 \boldsymbol{x_l} + d_3 \boldsymbol{\gamma} \boldsymbol{x_l} \le b_3, \tag{2.24d}$$

$$a_{4}\boldsymbol{x}_{\mathbf{l}} + a_{4}\Delta\boldsymbol{p} \le b_{4}, \qquad (2.24e)$$

$$\boldsymbol{x}_{\mathbf{l}} \geq \begin{bmatrix} b_2 & b_3 & (b_4 - d_4 \Delta \boldsymbol{p}) \end{bmatrix} \boldsymbol{\lambda},$$
 (2.24f)

$$\begin{bmatrix} a_2 & (a_3+d_3\boldsymbol{\gamma}) & a_4 \end{bmatrix} \boldsymbol{\lambda} = 1, \qquad (2.24g)$$

$$\boldsymbol{\lambda} \ge \boldsymbol{0}, \tag{2.24h}$$

In reality, we do not need the lower-level problem to determine the exact optimal value for  $x_1$ . We only need to guarantee that  $x_1$  is lower than the upper bounds specified by the constraints (2.24b) which were part of the leader in the bilevel problem. Since we already have an upper bound  $b_1$ for  $x_1$ , which in turn is an upper bound on the dual objective function, we can use the same bound  $b_1$ in (2.24f) to modify the strong duality constraint. The single-level optimization problem then reduces to

$$\max_{\Delta \boldsymbol{p}, \boldsymbol{\gamma}, \boldsymbol{x}_{\mathbf{l}}, \boldsymbol{\lambda}} \Delta \boldsymbol{p} \tag{2.25a}$$

s.t. 
$$a_2 x_1 \le b_2$$
, (2.25b)

$$a_3 \boldsymbol{x_l} + d_3 \boldsymbol{\gamma} \boldsymbol{x_l} \le b_3, \tag{2.25c}$$

$$a_4 \boldsymbol{x}_{\mathbf{l}} + d_4 \Delta \boldsymbol{p} \le b_4, \tag{2.25d}$$

$$b_1 \ge \begin{bmatrix} b_2 & b_3 & (b_4 - d_4 \Delta p) \end{bmatrix} \boldsymbol{\lambda},$$
 (2.25e)

$$\begin{bmatrix} a_2 & (a_3+d_3\gamma) & a_4 \end{bmatrix} \boldsymbol{\lambda} = 1, \qquad (2.25f)$$

$$\boldsymbol{\lambda} \ge 0. \tag{2.25g}$$

This single-level problem can now be solved using general purpose solvers. A drawback of the strong duality based reformulation is the presence of bilinear terms  $\gamma \cdot x_1$  in the primal follower problem constraint (2.25c),  $\Delta p \cdot \lambda$  in the strong duality constraint (2.25c) and  $\gamma \cdot \lambda$  in the dual equality constraint (2.25f) which make the problem non-convex and computationally challenging.

#### 2.3.2 Generalization to Multiple Follower Problems

A similar approach can be used to derive the single-level reduction of  $P^{\pm}$  which comprises of multiple follower problems. Note that as the number of follower problems increase, we get a large optimization problem with many bilinear terms.

For lower-level problems with objectives to find the minimum achievable voltage magnitude, the primal objective value is a lower bound instead of an upper bound in the strong duality constraint.

Since we already have a lower bound  $\underline{v}$  for  $x_{l}$ , it is possible to modify the strong duality constraint and formulate the single-level optimization problem similar to (2.25).

#### 2.3.3 Reformulation for Other Inverter Reactive Power Control Modes

We can choose other inverter reactive power control modes and formulate the bilevel problem and the single-level reformulated problem in the same way as described for the constant power factor mode. For the voltage-reactive power mode, the upper-level and lower-level variable vectors are defined by

$$\begin{aligned} \boldsymbol{X}_{\mathbf{u}} &:= \{\Delta \boldsymbol{p}^{+}, \Delta \boldsymbol{p}^{-}, \overline{\boldsymbol{q}}_{\mathbf{G},i}^{\phi} \forall_{\phi \in \Phi, i \in \mathscr{N}} \}, \\ \boldsymbol{X}_{\mathbf{l}} &:= \left\{ \{\boldsymbol{v}_{\mathbf{d}i}^{\phi}, \boldsymbol{v}_{\mathbf{q}i}^{\phi}, |\boldsymbol{v}_{i}^{\phi}|, \Delta \boldsymbol{p}_{\mathbf{G},i}^{\phi}, \Delta \boldsymbol{p}_{\mathbf{L},i}^{\phi}, \boldsymbol{q}_{\mathbf{G},i}^{\phi} \}_{\forall_{\phi \in \Phi, i \in \mathscr{N}}} \right\}. \end{aligned}$$

$$(2.26)$$

Recall that that we have bilinear terms in the inverter reactive power constraint, strong duality constraint as well as the dual equality constraints for the constant power factor mode. Similarly, for the voltage-reactive power mode, we get the same number of bilinear terms since the inverter constraints (2.17) are defined by product of upper-level and lower-level variables.

In case of the constant reactive power mode, the upper-level and lower-level variable vectors are given by

$$\begin{aligned} \boldsymbol{X}_{\mathbf{u}} &:= \{\Delta \boldsymbol{p}^{+}, \Delta \boldsymbol{p}^{-}, \boldsymbol{q}_{\mathbf{G},i}^{\phi} \forall_{\phi \in \Phi, i \in \mathcal{N}} \}, \\ \boldsymbol{X}_{\mathbf{l}} &:= \left\{ \{\boldsymbol{v}_{\mathbf{d}i}^{\phi}, \boldsymbol{v}_{\mathbf{q}i}^{\phi}, | \boldsymbol{v}_{i}^{\phi} |, \Delta \boldsymbol{p}_{\mathbf{G},i}^{\phi}, \Delta \boldsymbol{p}_{\mathbf{L},i}^{\phi}, \boldsymbol{q}_{\mathbf{L},i}^{\phi} \}_{\forall_{\phi \in \Phi, i \in \mathcal{N}}} \right\}. \end{aligned}$$

$$(2.27)$$

Different from the other two modes, the inverter reactive power constraints (2.16) do not consist of any bilinear terms and as a result, no bilinear terms are present in the dual equality constraints. However, the upper-level inverter variable  $q_{G,i}^{\phi}$  shows up in a number of other constraints. Because of this, we get a large number of bilinear terms in the strong duality constraint for the constant reactive power mode.

We will next describe an iterative solution approach used to efficiently handle the bilinear terms and solve the single-level problem (2.25).

#### 2.4 Iterative Solution Approach

As the number of leader problem variables or the number of follower problems in (2.19) increases, the number of bilinear terms in the single-level reformulated problem (2.25) also increase, making it computationally expensive to solve. One way to reduce the number of bilinear terms is to decrease the number of follower problems in the bilevel problem. Typically, the voltage magnitude violations in a distribution grid occur only at a few nodes, which in turn determine the aggregate power flexibility of the system. Instead of solving the bilevel problem with 12n number of follower problems, we can identify the nodes where voltage violations are most likely to occur and only

include the lower-level problems corresponding to these nodes in the bilevel optimization task. In this section, we use an iterative approach to identify such nodes and solve the problem efficiently. We will describe the method to solve for the single-level reformulated problem in this section. The iterative approach is summarized below and a more detailed explanation is provided in the next few subsections.

1) We solve a *worst-case problem* where we assume DSO determines the aggregate power flexibility limits  $\Delta p^+, \Delta p^-$  without any DER reactive power support i.e. aggregators are free to control even the reactive power injections of the DER inverters without considering grid constraints. This means that the inverter control variables become lower-level variables and  $\Delta p^+, \Delta p^-$  are the only upper-level variables. For all three inverter reactive power control modes, we have the same lower-level variables which are defined as

$$\boldsymbol{X}_{\mathbf{l}} := \left\{ \{ \boldsymbol{v}_{\mathbf{d}i}^{\phi}, \boldsymbol{v}_{\mathbf{q}i}^{\phi}, | \boldsymbol{v}_{i}^{\phi} |, \Delta \boldsymbol{p}_{\mathbf{G},i}^{\phi}, \Delta \boldsymbol{p}_{\mathbf{L},i}^{\phi}, \boldsymbol{q}_{\mathbf{G},i}^{\phi}, \boldsymbol{q}_{\mathbf{G},i}^{\phi} \}_{\forall_{\phi \in \Phi, i \in \mathcal{N}}} \right\}.$$
(2.28)

By solving the worst-case problem, we can determine the node where voltage violations (i.e. either overvoltage or undervoltage) are most likely to occur and as a result, provide the lowest aggregate power flexibility range.

- 2) We solve an *ideal case problem* which is already defined in  $P^{\pm}$  where the DSO determines the setpoints for DER reactive power control along with the aggregate power flexibility limits and rest of the DER power injections are controlled by the aggregators. Instead of including all 12*n* followers, we solve a relaxed version of the problem where we only include the lower-level problems corresponding to the single-phase connection which provides the lowest aggregate power flexibility range the worst-case problem in step 1).
- 3) We perform a *feasibility check* to ensure that the solution of the ideal case problem mitigates voltage violations at all the nodes. To do this, we use the aggregate power flexibility range and inverter setpoints determined by solving the ideal case problem and solve only the lower-level problems defined in  $P^{\pm}$  to find the worst-case voltage magnitudes.
- 4) If all voltage magnitudes are within limits, we terminate the iterative process. Otherwise, we go back to step 2) and solve the ideal case problem again by including an additional set of follower problems corresponding to the single-phase connection that did not pass the feasibility check.

### 2.4.1 Worst Case Problem

As mentioned previously, the inverter setpoints for the different inverter reactive power control modes are not considered as leader problem variables in the worst-case problem. Instead, we assume they are determined by the aggregator and may be equal to the values that would lead to worst-case conditions occurring in the grid. These worst-case values are determined by including them in the bilevel problem as lower-level variables. This is a straightforward approach for the constant reactive power mode where the inverter constraints (2.16) are linear. However, we have bilinear terms in the inverter constraints (2.15a), (2.17a) for constant power factor and voltage-

reactive power mode, respectively. To avoid these bilinear terms and ensure that we can solve the worst-case problem efficiently, we fix the inverter setpoints to one of their specified limits. For e.g., if the lower-level objective of the bilevel problem is to maximize voltage magnitude, the inverter setpoints for both reactive power modes are determined as follows:

- Constant power factor mode: We set  $\gamma_{G,i}^{\phi}$  for every inverter to its upper limit defined in (2.15b). This corresponds to the situation where all inverters are boosting the voltage profile in the network by injecting the full available reactive power leading to higher voltage magnitudes.
- *Voltage-reactive power mode*: We set  $\overline{q}_{\mathbf{G},i}^{\phi}$  to its lower limit defined in (2.17b). This represents the condition where inverters do not provide any reactive power support when voltage magnitude is close to or violating the upper limit  $\overline{v}$ .

Note that  $\Delta p^+, \Delta p^-$  are the only variables that are shared by all the lower-level problems in the worst-case problem. Apart from that, there are no other variables or constraints linking the 12*n* follower problems. This allows us to separate the problem into individual problems for each follower specifically. We first solve 12*n* worst-case problems with one leader and one follower each to obtain 6*n* aggregate power flexibility upper limits for the positive case (with  $\Delta p^- = 0$ ) and 6*n* aggregate power flexibility lower limits for the negative case (with  $\Delta p^+ = 0$ ). Next, we sort the 6*n* upper limits in ascending order and also sort the 6*n* lower limits in the descending order. The difference between the minimum upper limit and the maximum lower limit is the lowest range of aggregate power flexibility for the network. This value should be equal to the solution obtained by solving the worst-case problem with one leader and 12*n* followers because all the follower problems are independent of each other. The total computation time to solve the 12*n* smaller worst-case problems will be much lower than the time to solve one large worst-case problem with 12*n* followers, especially when parallel computing techniques are used. Note that each of the 12*n* small worst-case problems has only one bilinear term corresponding to the product of the dual variable and upper-level variable in the strong duality constraint.

### 2.4.2 Ideal Case Problem

We have multiple upper-level variables shared by the followers of the ideal case problem  $P^{\pm}$  as defined in (2.18). Hence, this problem cannot be broken down into multiple bilevel problems and solved independently like the worst-case problem. To reduce computational complexity, we initially only include the two lower-level problems (one each for positive and negative cases) corresponding to the two worst-case problems that determined the lowest aggregate power flexibility range in step 1). When we only have a few follower problems, the number of bilinear terms is much lower compared to the problem with 12n followers. As a result, the ideal case problem can be efficiently solved to obtain the aggregate power flexibility range and inverter setpoints. Note that the subset of lower-level problems included in the ideal case bilevel problem is sequentially increased, if needed, as described in the feasibility check section.

### 2.4.3 Feasibility Check

It is not guaranteed that the solution obtained by solving the ideal case problem mitigates voltage violations at all single-phase connections in the network since we only considered two followers in the bilevel problem corresponding to one single-phase connection. Hence, we perform a feasibility check by using the aggregate power flexibility range and inverter setpoints determined by the ideal case problem and solving the lower-level problems defined in  $P^{\pm}$  to determine worst-case voltage magnitudes. Recall that when we solve the worst-case problems in step 1) of the iterative approach, we sort the resulting 6n upper-limit and lower-limit ranges in ascending and descending orders, respectively. These two lists also provides us with an order for the single-phase connections, where the connections with maximum risk of voltage violations are at the top and single-phase connections with lowest risk of voltage violations are at the bottom of the list. We utilize this ordering when we are doing the feasibility check and solve the lower-level problems sequentially starting from the top of the lists. If we observe a voltage violation, we go back to step 2) and add the followers to the ideal case problem which correspond to the single-phase connection where the voltage violation occurred and solve the problem again. We repeat this process iteratively and terminate when the feasibility check determines that all worst-case voltage magnitudes are within limits. Note that the lower-level problems are fully linear with no bilinear terms since the inverter setpoints were already determined by the ideal case problem, making them easy to solve.

# 2.5 Numerical Results

We next perform different analysis to evaluate the performance of our proposed method:

- Section 2.5.1 investigates the **performance of the method on a small IEEE 13-node feeder**. More specifically, we want to evaluate if our solution approach is able to find aggregate power flexibility limits and DER inverter reactive power setpoints that mitigate voltage violations throughout the grid. We also check the **accuracy of the linear power flow approximation**.
- Section 2.5.2 investigates whether the proposed bilevel approach scales well for large, realistic distribution feeders. We provide these results for the PNNL taxonomic feeder: R2-12-47-2 [51].

For our analysis, we solve three optimization problems based on the three inverter reactive power control modes. The load power factor  $pf_{L,i}^{\phi}$  at node *i* connected to phase  $\phi$  is set to 0.95. For the solar PV inverters, we set the target power factor  $pf_{G,i}^{\phi} = 0.9$  for the constant power factor mode and the target power ratio  $\gamma_{G,i}^{\phi} = 0.48$  for the constant reactive power mode.

The bilevel optimization problem is implemented in Julia [52] and the optimization problem is solved using JuMP [53] and Gurobi solver [54]. All simulations were run on a Windows 10 PC with 3.00 GHz Intel Xeon processor and 16 GB RAM.

#### 2.5.1 Performance Evaluation using IEEE 13-Node Feeder

For our small test case, we use the modified IEEE 13-node feeder [55] illustrated in Fig. 2.2 with single-phase solar PV installations at seven nodes. The maximum apparent power rating of each single-phase solar PV inverter is 60 kVA. The PV penetration level, calculated as the ratio of total PV generation (in kW) to the total rated load (in kW), was chosen to be 45% for the forecasted operating point. The voltage limits are set to  $\underline{v} = 0.9$  p.u. and  $\overline{v} = 1.1$  p.u. The taps of the voltage regulator connecting three-phase nodes 650 and 630 are set to high values of (10,8,11) and as a result, the system is more prone to overvoltage conditions compared to undervoltage scenarios. For simplicity, we will only discuss results which focus on identifying aggregate power flexibility limits  $\Delta p^-$ ,  $\Delta p^+$  that mitigate overvoltages in the feeder (i.e. the lower-level problem objective is to maximize voltage magnitude). The maximum available aggregate power flexibility in the system, which can be determined by summing the individual bounds for load and PV inverter flexibility, is  $\pm 1.64$  MW.



Figure 2.2: Modified IEEE-13 node feeder.

### Worst-case Aggregate Power Flexibility Limits

The first step of our iterative solution approach is to solve the worst-case problem for each singlephase connection separately and identify the location where overvoltages are most likely going to occur. Fig. 2.3 shows the aggregate power flexibility limits obtained by solving the worst-case problem for the three inverter reactive power control modes. The orange triangles represent the solution obtained for the upper limit  $\Delta p^+$  and the blue triangles illustrate the solution obtained for the lower limit  $\Delta p^-$ . We observe for the constant power factor mode that the aggregate power flexibility obtained is different for different single-phase connections. The lower limit  $\Delta p^-$  is equal to -1.64 MW for most of the single-phase connections. Conversely, the upper limit  $\Delta p^-$  is much lower than 1.64 MW indicating that the overvoltages are most likely going to occur for the positive case compared to the negative case. For the reactive power mode, the aggregate power flexibility range is equal to zero for multiple single-phase connections. This is because the reactive power constraints are less restrictive compared to the constraints defined for the constant power factor mode. On the other hand, the results for the voltage-reactive power mode indicate that the reactive power constraints are most restrictive for this mode. Therefore, we see that the aggregate power flexibility range at almost all the single-phase connections is equal to the maximum available aggregate power flexibility.



Figure 2.3: IEEE-13 node feeder results. We show the aggregate power flexibility limits obtained by solving worst-case problem for every single-phase connection with different inverter reactive power control modes.

For all three inverter reactive power control modes, the smallest aggregate power flexibility range was obtained when the lower-level problem objective was to maximize voltage magnitude at phase b of node 675 (single-phase connection #46 in Fig. 2.3). To verify this, we solved a single worst-case problem with one leader problem and multiple follower problems and the obtained aggregate power flexibility limits are summarized in Table 2.3. We notice that the aggregate power flexibility range obtained for the different reactive power nodes is equal to the range obtained for single-phase connection #46 in Fig. 2.3. This verifies that both problems are equivalent.

Mode	$\Delta p^{-}$ (MW)	$\Delta p^+$ (MW)	Computation Time (sec)
Constant power factor	-0.18	0.03	4.6
Constant reactive power	0	0	3.1
Voltage-reactive power	-1.64	0.67	3.5

Table 2.3: IEEE-13 node feeder results for aggregate power flexibility obtained by solving a single Worst-case problem for different inverter reactive power control modes.

It is important to note that the worst-case problem with one follower has only two bilinear terms in the single-level reformulated problem which arises due to the product of the dual variable corresponding to aggregate power flexibility constraints and either of the two upper-level variable  $\Delta p^+$ ,  $\Delta p^-$ . Solvers such as Gurobi can handle this constraint efficiently using spatial branching if the number of bilinear terms are small and the average computation time to solve the worst-case problem with one follower is less than 0.07 seconds. Conversely, the computational time to solve the large worst-case problem with multiple follower problems is longer as seen in Table 2.3.

### **Ideal Case Aggregate Power Flexibility Limits**

We next solve the ideal case problem by starting with one lower-level problem corresponding to node 675 at phase *b*. After obtaining the inverter setpoints from the optimization, we perform a feasibility check to determine if more follower problems need to be added to the ideal case problem iteratively following the procedure described in Section 2.4. Table 2.4 summarizes the aggregate power flexibility limits obtained after termination of the iterative process. We observe that the aggregate power flexibility limits are same for all inverter reactive power control modes and they are equal to the maximum available aggregate power flexibility in the system. By comparing this with the conservative limits obtained by the worst-case problem in Table 2.3, we can conclude that it is important that the DSO is able to control the reactive power settings of the inverters to maximize the amount of flexibility offered by the DERs connected to the grid.

It is interesting to note that the feasibility check is passed after only a few iterations for all the problems. While only one follower is required for the problem with inverters operating in constant reactive power mode to pass the feasibility check, we observe for the constant power factor and voltage-reactive power modes that more follower problems are required. This might be because the single-level problem for these two modes consists of bilinear terms in the dual equality constraints. On the contrary, no bilinear terms are present in the dual equality constraints of the single-level problem for the constant reactive power mode. Furthermore, Table 2.4 show the computation time to solve the ideal case problem with multiple follower problems in the last iteration. We observe that the solve time is less than a second for all three inverter reactive power modes. This indicates that the iterative approach can solve the problem efficiently and scales well for larger distribution feeders.

Fig. 2.4 illustrates the inverter setpoints determined by the ideal case problem that ensure no overvoltages occur throughout the grid. For the constant power factor and constant reactive power modes, we observe that the reactive-power injections for most of the solar PV systems are negative.

Mode	$\Delta p^-$ (MW)	$\Delta p^+$ (MW)	No. of Iterations	Computation Time (sec)
Constant power factor	-1.64	1.64	3	0.10
Constant reactive power	-1.64	1.64	1	0.05
Voltage-reactive power	-1.64	1.64	3	0.50

Table 2.4: IEEE-13 node feeder results for aggregate power flexibility obtained by solving ideal case problem for different inverter reactive power control modes.

This is expected since the voltage profile of the feeder is already close to the upper limit due to the high regulator tap settings and hence, absorption of the reactive power will lead to lowering of the voltage magnitudes. For the voltage-reactive power mode, it can be seen that the maximum reactive power limit is chosen very close to the apparent power limit of the inverters (red dashed line) to ensure that the total reactive power absorbed by the solar PV systems is high thereby keeping the voltage magnitudes within limits.



Figure 2.4: IEEE-13 node feeder results obtained by solving ideal case problem. We show the setpoints determined for different inverter reactive power control modes. The red dashed lines are maximum and minimum limits on the upper-level variables.

#### **Linear Approximation Accuracy**

Recall that we linearized multiple constraints in the lower-level problem of our bilevel optimization task. To analyze the accuracy of the inverter setpoints obtained by solving the linear ideal case problem, we formulate the nonlinear counterpart of the lower-level problem by making the following changes:

- Quadratic inverter constraints are utilized instead of the circular linear constraints.
- We replace the first order Taylor approximation of the relationship between the voltage variables with the quadratic constraints.
- We use the nonlinear power flow equation in rectangular form instead of the fixed-point linear power flow model.

Consequently, the nonlinear lower-level problem with objective to find worst-case maximum voltage magnitudes for the positive case can be expressed as

$$|\overline{v}_i^{\phi}| = \max_{X_1} |v_i^{\phi}|, \ \forall_{\phi \in \Phi, i \in \mathcal{N}},$$
(2.29)

s.t. Load reactive power : (2.9),

Aggregate power flexibility : (2.11), (2.12),

Inverter reactive power modes : (2.15) or (2.16) or (2.17),

$$\begin{aligned} (p_{\mathbf{G},i}^{\phi} + \Delta p_{\mathbf{G},i}^{\phi})^{2} + (q_{\mathbf{G},i}^{\phi})^{2} &\leq |s_{\mathbf{G},i}^{\phi}|^{2}, \qquad \qquad \forall_{\phi \in \Phi, i \in \mathcal{N}}, \\ |v_{i}^{\phi}|^{2} &= (v_{\mathbf{d}i}^{\phi})^{2} + (v_{\mathbf{q}i}^{\phi})^{2}, \qquad \qquad \forall_{\phi \in \Phi, i \in \mathcal{N}}, \\ P_{i} &= \sum_{k \in \mathcal{N}_{0}} V_{\mathbf{d}i} \odot \left[ G_{ik} V_{\mathbf{d}k} - B_{ik} V_{\mathbf{q}k} \right] + \sum_{k \in \mathcal{N}_{0}} V_{\mathbf{q}i} \odot \left[ B_{ik} V_{\mathbf{d}k} + G_{ik} V_{\mathbf{q}k} \right], \qquad \forall_{i \in \mathcal{N}}, \\ Q_{i} &= \sum_{k \in \mathcal{N}_{0}} V_{\mathbf{d}i} \odot \left[ -B_{ik} V_{\mathbf{d}k} - G_{ik} V_{\mathbf{q}k} \right] + \sum_{k \in \mathcal{N}_{0}} V_{\mathbf{q}i} \odot \left[ G_{ik} V_{\mathbf{d}k} - B_{ik} V_{\mathbf{q}k} \right], \quad \forall_{i \in \mathcal{N}}. \end{aligned}$$

We can formulate a similar nonlinear problem for the negative case by just replacing the aggregate power flexibility constraints (2.12) with (2.13).

We substitute the aggregate power flexibility limits determined by the ideal case problem in (2.11) and the inverter setpoints in the constraints for the inverter reactive power modes. We then solve the optimization problem for all single-phase connections to check if the resulting voltage magnitudes are within the limits. We compare the voltage magnitudes obtained by solving (2.29) with the voltage magnitudes that are obtained by solving the linear lower-level problem of our bilevel task.

Fig. 2.5 shows the worst-case voltage magnitudes obtained by solving the nonlinear and linear versions of the problem. We observe that for all three inverter reactive power control modes, the setpoints determined by the ideal case problem keep the voltage magnitudes within the limits even we solve the nonlinear problem (2.29). We note that the approximation accuracy for the constant power factor and voltage-reactive power mode is high since the nonlinear voltage magnitudes (blue circles) and linear voltage magnitudes (orange crosses) are close to each other. On the other hand, the approximation accuracy is lower for the constant reactive power mode (maximum error  $\approx 0.006$ p.u.). This might be because the solution is further away from the initial operating point which was used to linearize the power flow equations in (2.7). However, we see that the fixed-point approximation is conservative for the constant reactive power mode since the voltage magnitudes determined by solving the nonlinear problem are generally lower than the voltage magnitudes obtained by the linear problem. Another noteworthy observation is that for a given single-phase connection, if we compare the worst-case voltage magnitudes obtained by all three modes, the results for the voltagereactive power mode exhibit the lowest magnitude. This indicates that voltage-reactive power mode is the most restrictive reactive power mode among the three modes and can be used effectively to mitigate voltage violations.



Figure 2.5: IEEE 13-node feeder results comparing the worst-case voltage magnitudes obtained by solving nonlinear and linear versions of the lower-level problem. The red dashed lines are the upper voltage magnitude limits.

### 2.5.2 Evaluating Scalability using Taxonomic Feeder-R2-12-47-2

We next investigate scalability of our solution method. For our larger test case, we choose the R2-12-47-2 feeder [51] shown in Fig. 2.6 which comprises of 820 single-phase connections and depicts a moderately populated suburban area with single family homes and light commercial loads [56]. Solar PV inverters are connected to all single-phase nodes in the feeder with each inverter rated at 5 KVA to achieve a PV penetration level of 20% of the total rated load. The voltage limits for this test case are tighter and set to  $\underline{v} = 0.95$  p.u. and  $\overline{v} = 1.05$  p.u. The maximum available aggregate power flexibility in the system is ±4.86 MW. In this case, we solve the bilevel problem to determine the aggregate power flexibility and inverter reactive power setpoints that mitigate both overvoltages and undervoltages in the feeder.



Figure 2.6: Modified R2-12-47-2 taxonomic feeder visualized using [1].

#### Worst-case Aggregate Power Flexibility Limits

We first look at the aggregate power flexibility limits obtained by solving the worst-case problem for every single-phase connection in the network. Different from the IEEE-13 node feeder results, we observe in Fig. 2.7 that all three inverter reactive power control modes provide an aggregate

power flexibility range of zero at multiple single-phase connections in the network. Furthermore, the average computation time to solve the worst-case problem at each single-phase connections is about 2 seconds.



Figure 2.7: R2-12-47-2 feeder results. We show the aggregate power flexibility limits obtained by solving worst-case problem for every single-phase connection with different inverter reactive power control modes.

Since there are multiple single-phase connections with aggregate power flexibility range of zero, we randomly pick node 135 connected to phase b (#571 in Fig. 2.7) whose corresponding lower-level problems will be included in the ideal case problem.

### **Ideal Case Aggregate Power Flexibility Limits**

We compare the results obtained after termination of the iterative approach where the ideal case problem is solved repeatedly until the feasibility check is passed. Table 2.5 summarizes the obtained aggregate power flexibility limits and computation time to solve the ideal case problem at the final iteration. Similar to the IEEE 13-node results, we observe that the aggregate power flexibility limits determined for all inverter reactive power control modes are equal to the maximum available aggregate power flexibility  $\pm 4.86$  MW in the system. It is interesting to note that the feasibility check passed after the first iteration for all three inverter reactive power control modes. In addition, the computation time to solve the ideal case problem with one follower was less than a minute for all three power modes. The higher solve time for the constant reactive power mode is mostly due to the higher number of bilinear terms in the reformulated problem. It is possible to employ piecewise McCormick envelopes to relax the bilinear terms [57] and further reduce computation time, but we defer this to future work.

The inverter setpoints determined by the ideal case problem for the different reactive power modes is shown in Fig. 2.8. Compared to the IEEE-13 node feeder results, most of the power ratio setpoints for the constant power factor mode are no longer close to the lower limit. Instead, we see that multiple inverters are injecting reactive power into the grid. This might be because the voltage profile of the feeder is not close to the upper limits and the lower PV penetration level reduces the probability of overvoltages and increases the risk of undervoltages. Hence, it is not required to absorb more reactive power from the network. The inverter setpoints obtained for the reactive

Mode	$\Delta p^{-}$ (MW)	$\Delta p^+$ (MW)	No. of Iterations	Computation Time (sec)
Constant power factor	-4.86	4.86	1	10
Constant reactive power	-4.86	4.86	1	45
Voltage-reactive power	-4.86	4.86	1	30

Table 2.5: R2-12-47-2 feeder results for aggregate power flexibility obtained by solving ideal case problem for different inverter reactive power control modes.

power mode are closer to zero compared to the lower limit which is similar to the reactive power setpoints obtained for the IEEE-13 node feeder in Fig. 2.4(b). On the other hand, for the voltage-reactive power mode, the maximum reactive power setpoints are closer to the lower limit as opposed to the IEEE-13 node results shown in Fig. 2.4(c) where the setpoints are near the upper limit. This verifies the conclusion made by looking at results for the constant power factor mode that the reactive power absorption is not required to be very high for this test case.



Figure 2.8: R2-12-47-2 feeder results obtained by solving ideal case problem. We show the setpoints determined for different inverter reactive power control modes. The red dashed lines are maximum and minimum limits on the upper-level variables.

One of the assumptions in the previous chapter is that measurement data (i.e. power injections, voltage phasors) is available at all nodes in the network. As a result, we assume to have full knowledge about the current operating point of the system. However, as has been discussed in other parts of this report, the real-time measurements might not be available at all (or any) points in the grid due to high cost of monitoring equipment, communication burden and privacy concerns of the endconsumers. In this section, we extend the bilevel formulation described above to include the case where the current operating point is not known or partially known. To consider this limitation while also maintaining security of operations, we treat all unknown measurements (and in particular, all unknown nodal power injections) as optimization variables in the follower of the bilevel problem. This will result in a more conservative worst-case operating condition compared to the previous case where we assumed that these quantities were fixed to their current value.

#### 3.1 System Modeling

In this section, we assume that measurements are not available at any node  $i \in \mathcal{N}$  and the voltage variables are denoted using  $\hat{V}_{\mathbf{d}i}, \hat{V}_{\mathbf{q}i}, |\hat{V}_i| \in \mathbb{R}^3$ , respectively. The active and reactive power injections at node  $i \in \mathcal{N}$  connected to phase  $\phi$  at the current operating point are given by

$$\hat{p}_i^{\phi} = \hat{p}_{\mathbf{G},i}^{\phi} - \hat{p}_{\mathbf{L},i}^{\phi}, \qquad (3.1a)$$

$$\hat{q}_{i}^{\phi} = \hat{q}_{\mathbf{G},i}^{\phi} - \hat{q}_{\mathbf{L},i}^{\phi},$$
 (3.1b)

where  $\hat{p}_{\mathbf{G},i}^{\phi}, \hat{q}_{\mathbf{G},i}^{\phi}$  are the respective active and reactive power generation variables for solar PV inverters;  $\hat{p}_{\mathbf{L},i}^{\phi}, \hat{q}_{\mathbf{L},i}^{\phi}$  represent the active and reactive load demand variables, respectively.

### 3.1.1 Load Modelling

For a load at node  $i \in \mathcal{N}$  connected to phase  $\phi \in \Phi$ , the limits for active power demand become

$$\underline{p}_{\mathrm{L},i}^{\phi} \le \hat{p}_{\mathrm{L},i}^{\phi} \le \overline{p}_{\mathrm{L},i}^{\phi}, \tag{3.2a}$$

$$\underline{p}_{\mathbf{L},i}^{\phi} \leq \hat{p}_{\mathbf{L},i}^{\phi} + \Delta p_{\mathbf{L},i}^{\phi} \leq \overline{p}_{\mathbf{L},i}^{\phi}.$$
(3.2b)

Assuming that loads operate in constant power factor  $pf_{L,i}^{\phi}$ , the reactive power consumption is

$$\hat{q}_{\mathbf{L},i}^{\phi} = \frac{\sqrt{1 - (pf_{\mathbf{L},i}^{\phi})^{2}}}{pf_{\mathbf{L},i}^{\phi}} \cdot \hat{p}_{\mathbf{L},i}^{\phi}, \qquad (3.3a)$$

$$\boldsymbol{q}_{\mathbf{L},i}^{\phi} = \frac{\sqrt{1 - \left(pf_{\mathbf{L},i}^{\phi}\right)^{2}}}{pf_{\mathbf{L},i}^{\phi}} \cdot \left(\hat{\boldsymbol{p}}_{\mathbf{L},i}^{\phi} + \Delta \boldsymbol{p}_{\mathbf{L},i}^{\phi}\right),$$
(3.3b)

where  $\hat{q}^{\phi}_{\mathbf{L},i}$  is the reactive power consumption at the current operating point.

### 3.1.2 Modeling of Solar PV Active Power Generation

The active power generation limits defined in (2.10) can be reformulated to include the power generation variable  $\hat{p}_{\mathbf{G},i}^{\phi}$  corresponding to the current operating point to get

$$\underline{p}_{\mathbf{G},i}^{\phi} \leq \hat{\boldsymbol{p}}_{\mathbf{G},i}^{\phi} \leq \overline{p}_{\mathbf{G},i}^{\phi}, \tag{3.4a}$$

$$\underline{p}_{\mathbf{G},i}^{\phi} \leq \hat{\boldsymbol{p}}_{\mathbf{G},i}^{\phi} + \Delta \boldsymbol{p}_{\mathbf{G},i}^{\phi} \leq \overline{p}_{\mathbf{G},i}^{\phi}.$$
(3.4b)

# 3.1.3 Inverter Constraints

The inverter limits defined in (2.14) can be reformulated to include the power generation variables  $\hat{p}_{\mathbf{G},i}^{\phi}, \hat{q}_{\mathbf{G},i}^{\phi}$  to get

$$0 \le \hat{\boldsymbol{p}}_{\mathbf{G},i}^{\phi} \le |\boldsymbol{s}_{\mathbf{G},i}^{\phi}|, \tag{3.5a}$$

$$-|s_{\mathbf{G},i}^{\phi}| \leq \hat{q}_{\mathbf{G},i}^{\phi} \leq |s_{\mathbf{G},i}^{\phi}|, \tag{3.5b}$$

$$\hat{\boldsymbol{p}}^{\phi}_{\mathbf{G},i} + \hat{\boldsymbol{q}}^{\phi}_{\mathbf{G},i} \le \sqrt{2} \cdot |\boldsymbol{s}^{\phi}_{\mathbf{G},i}|, \qquad (3.5c)$$

$$\hat{\boldsymbol{p}}_{\mathbf{G},i}^{\phi} - \hat{\boldsymbol{q}}_{\mathbf{G},i}^{\phi} \le \sqrt{2} \cdot |\boldsymbol{s}_{\mathbf{G},i}^{\phi}|, \qquad (3.5d)$$

$$0 \le \hat{\boldsymbol{p}}_{\mathbf{G},i}^{\phi} + \Delta \boldsymbol{p}_{\mathbf{G},i}^{\phi} \le |\boldsymbol{s}_{\mathbf{G},i}^{\phi}|, \qquad (3.5e)$$

$$-|s_{\mathbf{G},i}^{\varphi}| \le \boldsymbol{q}_{\mathbf{G},i}^{\varphi} \le |s_{\mathbf{G},i}^{\varphi}|, \tag{3.5f}$$

$$\hat{\boldsymbol{p}}_{\mathbf{G},i}^{\phi} + \Delta \boldsymbol{p}_{\mathbf{G},i}^{\phi} + \boldsymbol{q}_{\mathbf{G},i}^{\phi} \le \sqrt{2} \cdot |\boldsymbol{s}_{\mathbf{G},i}^{\phi}|, \qquad (3.5g)$$

$$\hat{\boldsymbol{p}}_{\mathbf{G},i}^{\phi} + \Delta \boldsymbol{p}_{\mathbf{G},i}^{\phi} - \boldsymbol{q}_{\mathbf{G},i}^{\phi} \le \sqrt{2} \cdot |\boldsymbol{s}_{\mathbf{G},i}^{\phi}|.$$
(3.5h)

For the constant power mode, we can express the reactive power injections using

$$\hat{\boldsymbol{q}}_{\mathbf{G},i}^{\phi} = \hat{\boldsymbol{\gamma}}_{\mathbf{G},i}^{\phi} \cdot \hat{\boldsymbol{p}}_{\mathbf{G},i}^{\phi}, \qquad (3.6a)$$

$$\boldsymbol{q}_{\mathbf{G},i}^{\phi} = \boldsymbol{\gamma}_{\mathbf{G},i}^{\phi} \cdot \left( \hat{\boldsymbol{p}}_{\mathbf{G},i}^{\phi} + \Delta \boldsymbol{p}_{\mathbf{G},i}^{\phi} \right).$$
(3.6b)

where  $\hat{\gamma}_{G,i}^{\phi}$  represents the power ratio of inverter at the current operating point. We assume that  $\hat{\gamma}_{G,i}^{\phi}$  is specified by the DSO and hence, the value is already known. Note that the limits for  $\gamma_{G,i}^{\phi}$  were already defined in (2.15b).

For the constant reactive power mode, we again assume that the reactive power injections at the current operating point are known because they are controlled by the DSO and denoted by  $\hat{q}_{G,i}^{\phi}$ . So, we replace the limits defined in (2.16) by

$$\hat{q}^{\phi}_{\mathbf{G},i} = \hat{q}^{\phi}_{\mathbf{G},i},$$
 (3.7a)

$$-\gamma_{\mathbf{G},i}^{\phi} \cdot \left(\hat{\boldsymbol{p}}_{\mathbf{G},i}^{\phi} + \Delta \boldsymbol{p}_{\mathbf{G},i}^{\phi}\right) \leq \boldsymbol{q}_{\mathbf{G},i}^{\phi} \leq \gamma_{\mathbf{G},i}^{\phi} \cdot \left(\hat{\boldsymbol{p}}_{\mathbf{G},i}^{\phi} + \Delta \boldsymbol{p}_{\mathbf{G},i}^{\phi}\right).$$
(3.7b)

For the voltage-reactive power mode, we assume that DSO has information about the maximum available reactive power at the current operating point  $\bar{\hat{q}}_{G,i}^{\phi}$ . We still need to include additional reactive power constraints which are expressed as

$$\hat{\boldsymbol{q}}_{\mathbf{G},i}^{\phi} = \overline{\hat{\boldsymbol{q}}}_{\mathbf{G},i}^{\phi} - 2\overline{\hat{\boldsymbol{q}}}_{\mathbf{G},i}^{\phi} \cdot \left(\frac{|\hat{\boldsymbol{v}}_{i}^{\phi}| - \underline{\boldsymbol{v}}}{\overline{\boldsymbol{v}} - \underline{\boldsymbol{v}}}\right).$$
(3.8)

Here, the voltage magnitude at current operating point  $|\hat{v}_i^{\phi}|$  is constrained by

$$\underline{v} \le |\hat{v}_i^{\varphi}| \le \overline{v}. \tag{3.9}$$

Furthermore, we express the relationship between  $|\hat{v}_i^{\phi}|$  and the rectangular form voltage variables at the current operating point  $\hat{v}_{\mathbf{d}i}^{\phi}, \hat{v}_{\mathbf{q}i}^{\phi}$  similar to (2.3) using

$$(v_{di}^{\phi})^{2} + (v_{qi}^{\phi})^{2} + 2v_{di}^{\phi}\hat{v}_{di}^{\phi} + 2v_{qi}^{\phi}\hat{v}_{qi}^{\phi} = |v_{i}^{\phi}|^{2} + 2|v_{i}^{\phi}||\hat{v}_{i}^{\phi}|, \qquad (3.10)$$

where  $v_{di}^{\phi}$ ,  $v_{qi}^{\phi}$  are the real and imaginary components of the initial voltage estimate, respectively. Note that in the previous section, we used the current operating point as the initial voltage estimate. In this case, we use the no-load solution defined in (2.6) as the initial voltage estimate for both (2.3) and (3.10).

### 3.1.4 Power Flow

Following (2.7) and using the no-load solution as the initial voltage estimate, we enforce the power balance constraints at the current operating point using

$$\hat{\boldsymbol{V}}_{\mathbf{d}} = \Re\left\{Z_1\right\} + \Re\left\{Z_2\right\} \cdot \hat{\boldsymbol{P}}_{\mathbf{Y}} + \Im\left\{Z_2\right\} \cdot \hat{\boldsymbol{Q}}_{\mathbf{Y}}, \qquad (3.11a)$$

$$\hat{\boldsymbol{V}}_{\boldsymbol{q}} = \Im\left\{\boldsymbol{Z}_{1}\right\} + \Im\left\{\boldsymbol{Z}_{2}\right\} \cdot \hat{\boldsymbol{P}}_{\boldsymbol{Y}} - \Re\left\{\boldsymbol{Z}_{2}\right\} \cdot \hat{\boldsymbol{Q}}_{\boldsymbol{Y}}, \qquad (3.11b)$$

where  $\hat{V}_{\mathbf{d}}$ ,  $\hat{V}_{\mathbf{q}} \in \mathbb{R}^{3n}$  are vectors of the real and imaginary voltage components of the current operating point, respectively;  $\hat{P}_{\mathbf{Y}}$ ,  $\hat{Q}_{\mathbf{Y}} \in \mathbb{R}^{3n}$  denote the respective active and reactive power injections at the current operating point. Note that (3.11) also requires an initial voltage estimate to calculate  $Z_1, Z_2$  and we use the same no-load voltage solution for this.

#### 3.1.5 Bilevel Problem

In this section, we discuss the bilevel problem formulation where inverters are operating in constant power factor mode. We can follow the same procedure to formulate bilevel problems for the other inverter reactive power control modes. The upper-level and lower-level variables for the constant power factor mode are given by

$$\begin{aligned} \boldsymbol{X}_{\mathbf{u}} &:= \{\Delta \boldsymbol{p}^{+}, \Delta \boldsymbol{p}^{-}, \boldsymbol{\gamma}_{\mathbf{G},i}^{\phi} \forall_{\phi \in \Phi, i \in \mathcal{N}} \}, \\ \boldsymbol{X}_{\mathbf{l}} &:= \left\{ \{\boldsymbol{v}_{\mathbf{d}i}^{\phi}, \boldsymbol{v}_{\mathbf{q}i}^{\phi}, | \boldsymbol{v}_{i}^{\phi} |, \Delta \boldsymbol{p}_{\mathbf{G},i}^{\phi}, \Delta \boldsymbol{p}_{\mathbf{L},i}^{\phi}, \boldsymbol{q}_{\mathbf{G},i}^{\phi} \}_{\forall_{\phi \in \Phi, i \in \mathcal{N}}} \right\}, \\ \hat{\boldsymbol{X}}_{\mathbf{l}} &:= \left\{ \{\hat{\boldsymbol{v}}_{\mathbf{d}i}^{\phi}, \hat{\boldsymbol{v}}_{\mathbf{q}i}^{\phi}, | \hat{\boldsymbol{v}}_{k}^{\phi} |, \hat{\boldsymbol{p}}_{\mathbf{G},i}^{\phi}, \hat{\boldsymbol{p}}_{\mathbf{G},i}^{\phi}, \hat{\boldsymbol{q}}_{\mathbf{G},i}^{\phi} \}_{\forall_{\phi \in \Phi, i \in \mathcal{N}}} \right\}. \end{aligned}$$
(3.12)

Note that the additional lower-level problem variables  $\hat{X}_{l}$  correspond to the unknown current operating point. Furthermore, we consider the four scenarios as summarized in Table 2.2 and define four sets of lower-level variables  $\underline{X}_{l}^{+}, \underline{X}_{l}^{-}, \overline{X}_{l}^{+}, \overline{X}_{l}^{-}$ . However, the current operating point does not change for the four scenarios and as a result, we only need one set for  $\hat{X}_{l}$ .

Similar to the procedure followed in Section 2.2, we define an extra set of lower-level system constraints which depend on  $\hat{X}_{l}$  using

$$\mathbb{L}(\hat{X}_{l}) := \begin{cases} \text{Voltage constraints (3.10),} \\ \text{Load reactive power constraints (3.3),} \\ \text{Power flow (3.11).} \end{cases}$$

The four sets of inverter constraints depend on  $\hat{X}_{\mathbf{l}}$  and can be represented by

$$\mathbb{I}(\boldsymbol{\gamma}_{\mathbf{G},i}^{\phi}, \underline{X}_{\mathbf{l}}^{+}, \hat{X}_{\mathbf{l}}), \mathbb{I}(\boldsymbol{\gamma}_{\mathbf{G},i}^{\phi}, \underline{X}_{\mathbf{l}}^{-}, \hat{X}_{\mathbf{l}}), \\ \mathbb{I}(\boldsymbol{\gamma}_{\mathbf{G},i}^{\phi}, \overline{X}_{\mathbf{l}}^{+}, \hat{X}_{\mathbf{l}}), \mathbb{I}(\boldsymbol{\gamma}_{\mathbf{G},i}^{\phi}, \overline{X}_{\mathbf{l}}^{-}, \hat{X}_{\mathbf{l}}) \right\} := \text{Inverter constraints (3.5), (3.6).}$$

It is important to note here that the lower-level variable vector  $\hat{X}_{l}$  is shared by all four sets.

The sets of active power flexibility constraints for the positive and negative case do not depend on  $\hat{X}_{l}$ . So, we can use the same sets defined in Section 2.2 along with the above defined sets and formulate a bilevel problem similar to P<sup>±</sup>. The single-level reformulation of the bilevel problem can be performed using the steps described in Section 2.3.

We employ the same iterative solution approach detailed in Section 2.4 to solve the single-level optimization problem efficiently. It is important to note here that in step 1) of the iterative method, we solve 12n worst-case problems with one leader and one follower each and determine the overall aggregate power flexibility using the obtained solutions. Since the lower-level variables  $\hat{X}_1$  are also shared by the follower problems along with upper-level variables  $\Delta p^+, \Delta p^-$ , the resulting aggregate power flexibility will be more conservative than the solution determined by solving one

large worst-case problem with 12*n* followers (i.e. they are no longer equal to each other). Since the solutions are more conservative, we can still use them to identify the lower-level problems that need to be included in the ideal case problem in step 2) of the iterative approach.

#### 3.2 Numerical Results

To investigate the effect of not knowing the current operating point, we use the same IEEE 13node feeder described in Section 2.5.1 and focus on identifying aggregate power flexibility limits  $\Delta p^-, \Delta p^+$  that mitigate overvoltages in the feeder. As mentioned previously, the maximum available aggregate power flexibility in the system is  $\pm 1.64$  MW.

Recall that the DER setpoints at the current operating point are assumed to be known since they are controlled by the DSO. For simplicity, we assume in this case study that the setpoints are equal to zero, i.e.  $\hat{\gamma}_{G,i}^{\phi} = 0$  for the constant power factor mode,  $\hat{q}_{G,i}^{\phi} = 0$  for the constant reactive power mode and  $\bar{q}_{G,i}^{\phi} = 0$  for the voltage-reactive power mode. This means that the reactive power injections of all the DER inverters at the current operating point is zero for all three reactive power control modes.

### 3.2.1 Worst-case Aggregate Power Flexibility Limits

Fig. 3.1 illustrates the range of aggregate power flexibility obtained by solving the worst-case problem for the three inverter reactive power control modes. Different from the results in Fig. 2.3, we observe that the aggregate power flexibility range is extremely conservative since the current system state is not known and as a result, the problem finds the worst-case current operating point. Similar to the results in Section 2.5.1, we again choose to start solving the ideal case problem by including the followers corresponding to phase b of node 675 since the aggregate power flexibility range is zero for all three inverter reactive power control modes.



Figure 3.1: IEEE-13 node feeder results with limited measurements. We show the aggregate power flexibility limits obtained by solving worst-case problem for every single-phase connection with different inverter reactive power control modes.

### 3.2.2 Ideal Case Aggregate Power Flexibility Limits

Table 3.1 summarizes the final results obtained by solving the ideal case problem iteratively. We observe that it is still possible to obtain results with the maximum available aggregate power flexibility in the system. However, we see that the feasibility check is passed after a higher number of iterations compared to the results shown in Table 2.4. Furthermore, the computation time to solve the ideal case problem with multiple follower problems in the last iteration is higher due to the larger number of variables in the optimization problem.

Mode	$\Delta p^{-}$ (MW)	$\Delta p^+$ (MW)	No. of Iterations	Computation Time (sec)
Constant power factor	-1.64	1.64	5	2.9
Constant reactive power	-1.64	1.64	6	1.5
Voltage-reactive power	-1.64	1.64	3	2.2

Table 3.1: IEEE-13 node feeder results for aggregate power flexibility obtained by solving ideal case problem with limited measurements.

Fig. 3.2 illustrates the inverter setpoints determined by solving the ideal case problem. We observe that the reactive-power injections for most of the solar PV systems are negative and the set-points are similar to the results shown in Fig. 2.4. Recall that this happens because the voltage profile of the feeder is already close to the upper limit due to the high voltage regulator tap settings and as a result, the reactive power is absorbed by PV inverters to lower the voltage profile.



Figure 3.2: IEEE-13 node feeder results obtained by solving ideal case problem with limited measurements. We show the setpoints determined for different inverter reactive power control modes. The red dashed lines are maximum and minimum limits on the upper-level variables.

The main takeaway from the results shown above is that it is possible to identify reactive power setpoints that allow us to use the maximum available aggregate power flexibility in the system even when limited measurements are available. However, this cannot be generalized for all cases. The IEEE 13-node feeder used here is a smaller test case with a PV penetration level less that 50%. For

larger and more realistically sized distribution feeders with higher PV penetration levels, it might be challenging to identify such reactive power setpoints that allows us to harness the full collective flexibility of the DERs.

### 3.2.3 Linear Approximation Accuracy

We finally investigate the solution quality by solving the linear and nonlinear versions of the lowerlevel problem using the inverter setpoints determined above. Fig. 3.3 shows the worst-case voltage magnitudes at all single-phase connections when solving the nonlinear and linear problems with limited measurements. Similar to the results shown in Fig. 2.5, we see that for all three inverter reactive power control modes, the setpoints determined by the ideal case problem maintain the voltage magnitudes within the limits even for the nonlinear case. Even though we do not use the current operating point as the initial voltage estimate in our linear power flow equations and instead use a no-load voltage estimate, the approximation accuracy is high (maximum error  $\approx 0.011$  p.u.) for all three inverter reactive power control modes. Different from the results in Fig. 2.5, we observe that the fixed-point approximation is no longer conservative and the voltage magnitudes determined by solving the nonlinear problem are not always lower than the voltage magnitudes obtained by the linear problem.



Figure 3.3: IEEE 13-node feeder results comparing the worst-case voltage magnitudes obtained by solving nonlinear and linear versions of the lower-level problem with limited measurements. The red dashed lines are the upper voltage magnitude limits.

# 4. Conclusions

This part of our report provides a new method for coordination between transmission systems, distribution grids and DER aggregators. The main goal is to identify the aggregate power flexibility range provided to transmission systems without causing any constraint violations in distribution grids. We formulate a bilevel optimization task where the upper-level problem maximizes the aggregate power flexibility and determines the DER inverter reactive power setpoints while the lower-level problem determines the worst-case disaggregation strategy.

Our case studies demonstrate that it is important to allow the DSO to choose the DER inverter reactive power setpoints in order to utilize the maximum available flexibility in the system. These setpoints will ensure that the grid is secure even during worst-case conditions. The results for the IEEE 13-bus feeder showed that we are able to obtain high-quality solutions for the inverter setpoints by solving the single-level, strong duality based reformulated problem. We also observed that the proposed approach performs well even when limited measurements are available. Our results on the larger taxonomic feeder demonstrate that the proposed approach scales well and can be applied to large, realistic distribution feeders.

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