VaR Constrained Hedging of Fixed Price Load-Following Obligations in Competitive Electricity Markets *

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Abstract

Load serving entities providing electricity to regulated customers have an obligation to serve load that is subject to systematic and random fluctuations at fixed prices. In some jurisdictions like New Jersey, such obligations are auctioned off annually to third parties that commit to serve a fixed percentage of the fluctuating load at a fixed energy price. In either case the entity holding the load following obligation is exposed to the load variation and to a volatile wholesale spot market price which is correlated with the load level. Such double exposure to price and volume results in a net revenue exposure that is quadratic in price and cannot be adequately hedged with simple forward contracts whose payoff is linear in price. A fixed quantity forward contract cover, is likely to be short when the spot price is high and long when the spot price is low. In this paper we develop a self-financed hedging portfolio consisting of a risk free bond, a forward contract and a spectrum of call and put options with different strike prices. A popular portfolio design criterion is the maximization of expected hedged profits subject to a value at risk (VaR) constraint. Unfortunately, that criteria is

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difficult to implement directly due to the complicated form of the VaR constraint. We show, however, that under plausible distributional assumptions, the optimal VaR constrained portfolio is on the efficient Mean-Variance frontier. Hence, we propose an approximation method that restricts the search for the optimal VaR constrained portfolio to that efficient frontier. The proposed approach is particularly attractive when the Mean-Variance efficient frontier can be represented analytically, as is the case, when the load and logarithm of price follow a bivariate normal distribution. We illustrate the results with a numerical example.

Key Words: Energy Risk, Competitive Electricity Markets, Volumetric Hedging, Incomplete Markets

1 Introduction

Electricity is traded in the wholesale markets by numerous market participants such as generators, load-serving entities (LSEs)[1], and marketers at the prices determined by supply and demand equilibrium. Electricity market participants are exposed to risks in their net earnings due to uncertain wholesale market prices.

Electricity market prices are infamous for extremely high volatility. During the summer of 1998, wholesale power prices in the Midwest of the U.S. surged to a stunning amount of $7,000/MWh from the normal price range of $30 $\sim$ $60/MWh causing the defaults of two power marketers on the east coast. In February 2004, persistent high prices in Texas during an ice storm that lasted three days led to the bankruptcy of a retail energy provider that was exposed to spot market prices. More recently in January and in June 2007 The Australian Electricity Market experienced several events were prices rose to their maximum allowed level of 10,000 Australian Dollar per MWh and in Texas on March 3, 2008, two days after price caps on electricity were raised to $2250/MWh, electricity prices reached that high level due to a sudden drop of 1500MW in wind power generation.

In California during the 2000/2001 electricity crisis wholesale spot prices rose sharply and persisted around $500/MWh. The devastating economic

[1] Load-serving entities are companies who procure electricity from wholesale electricity markets to serve their customer's electricity needs.

consequences of that crisis were largely attributed to the fact that the major utilities, who were forced to sell power to their customers at low fixed prices set by the regulator, were not properly hedged through long-term supply contracts. Such expensive lessons have raised the awareness of market participants to the importance and necessity of risk management practices in the competitive electricity market.

The evident price risk in the competitive electricity market has fueled the emergence of risk management practices such as forward contracting and various hedging strategies. However, hedging strategies that only concern price risks for a fixed amount of volume cannot fully hedge market risks faced by LSEs, who are obligated to serve the uncertain electricity demand of their customers. The volumetric risk - caused by demand uncertainty - is a crucial risk factor faced by LSEs that must serve their customers at regulated fixed retail prices which can only be adjusted infrequently.

In some jurisdictions like New Jersey, LSEs auction off their default load serving obligations to private entities who assume the obligation to serve a fixed percentage slice of the total load at predetermined fixed retail prices, set through an annual auction [LS04]. Such entities assume the exposure to joint price and quantity risk which they typically cover through a mix of owned generation capacity, procurement of physical supply contracts and through various financial hedging strategies.

Volumetric risks in the electricity markets can be very severe due to the adverse movements of the wholesale price and demand both of which are affected by weather conditions; for instance, the sales volume is small when the profit margin is high, while it is large when the margin is low or even negative. This is due to the price inelasticity of demand and the resulting strong positive correlation between spot price and demand.

When such volumetric risk is involved, a company should hedge against fluctuations in total cost, i.e., quantity times price but unfortunately, there are no simple direct market instruments that would enable such hedging and more complex hedging strategies are needed. This paper investigates such hedging strategies designed to mitigate both price and volumetric risks faced by LSEs or default service providers holding fixed price load following obligations.

Our earlier paper [OOD06] was devoted to constructing the optimal static portfolio which consists of electricity derivatives such as forwards and calls and puts of different strikes. Specifically, we obtained the optimal hedging strategy that uses electricity derivatives to hedge price and volumetric risks by maximizing the expected utility of the hedged profit. When such a portfolio is held by an LSE, the call options with strikes being below the spot price will be exercised so that the quantity corresponding to options being exercised is procured at the strike prices. Using this strategy, the LSE can set an increasing price limit on incremental load by paying the premiums for the options. This strategy is not only effective in managing quantity risk but was also suggested in the market design literature such as Chao and Wilson [CW04], Oren [Ore05], and Willems [Wil06] as means to achieve resource adequacy, mitigate market power, and reduce spot price volatility.

In this paper, we extend our previous work by focusing on optimal self-financed hedging portfolios that maximizes expected net hedged cashflow (profit) subject to a Value-at-Risk (VaR) constraint on that quantity.

The LSE's hedging problem of price and quantity risk under the VaR criteria has been considered by Woo et al. [WKH04], Wagner et al. [WSI03], and Kleindorfer and Li [KL05]. The VaR, which is defined as a maximum possible loss with $(1-\gamma)$ percent confidence, is a widely-used risk measure in practice which has become a standard tool in risk management. However, the optimization problems with the VaR risk measure are hard to solve analytically without very restrictive assumptions, especially when price and quantity risks are considered.

Woo et al. [WKH04] solved for a forward position q in order to minimize the expected procurement cost $PQ+(F-P)q$ subject to the VaR constraint where P, Q, and F are spot price, demand, and forward price, respectively. They solved the problem heuristically using a simple spreadsheet by setting possible hedge ratios first, and examining the risk exposure on total cost. Their normal distribution assumption on the procurement cost simplified the calculation of the VaR measure.

More rigorous optimization was performed by Wagner et al. [WSI03] to determine the amounts of monthly forward contracts to be purchased for the upcoming several months. For an LSE who has to supply power at a fixed rate, they provided a simulation-based algorithm to solve the VaR-constrained problem, the problem of maximizing the expected hedged profit under the VaR constraint. However, their method is inefficient because one has to evaluate VaR for all possible combinations on the number of different forward contracts.

Handling VaR analytically usually requires a normality assumption on the hedged cash flow as in Ahn et al [ABRW99]. However, this normality assumption is not suitable for problems where the cash flow distribution is fat-tailed, like an LSEs's cash flow.

Kleindorfer and Li [KL05] found a more relaxed assumption than normality while still maintaining the tractability of the normal distribution. Basically, when VaR is monotone in the variance, multi-period VaR-constrained problems were shown to be equivalent to mean-variance problems. Moreover, they solved the mean-variance problem that included various types of contracts including options over the planning horizon by transforming them into solvable quadratic programs. Kleindorfer and Li obtain the market prices of derivatives and the mean and covariances of the wholesale electricity price, demand, and option payoffs from a simulation package to find the optimal number of derivative contracts.

In this paper, we seek a self financed, hedging portfolio that maximizes the expected profit subject to price and volumetric risk with a VaR constraint (Section 2). In our formulation we represent the hedging portfolio, as a general self-financed exotic option with a nonlinear payoff contingent on the price of electricity. Once we obtain the desired payoff function we replicate it with a portfolio consisting of bonds, at the money forwards, along with a spectrum of calls and puts, with a continuum of strike prices.

We first motivate our proposed approximation method for the VaR constrained optimal portfolio by identifying conditions, in the spirit of Kleindorfer and Li, under which the solution is on the efficient frontier with respect to a mean-variance portfolio selection criterion. This property holds, in the case of a normal, student-t, and Weibull distributions and more generally for distributions where the VaR is a function of the mean and standard deviations, which is monotonically increasing in standard deviation and non-increasing in the mean (Section 3).

We exploit this property to approximate the optimal VaR constrained hedging portfolio by restricting the search to hedging portfolios on the efficient mean-variance frontier under particular distributional assumptions (Section 4). Unfortunately, we cannot prove that the hedged profit distribution satisfies the required monotonicity properties so the solution we obtain might be suboptimal. The search of suboptimal solutions to the VaR constrained problem is not uncommon and can be justified from various perspectives. For instance, we can exploit the fact that the Chebyshev's upper bound on the VaR of any distribution is a function of the mean and standard deviations, which is monotonically increasing in standard deviation and non-increasing in the mean. Hence, if we tighten the VaR constraint by replacing it with a constraint on the Chebyshev bound, the optimal solution to the more constrained problem (which is suboptimal for the original VaR constrained problem) lies on the efficient mean-variance frontier [AB02]. One could also argue that the tightness of the Chebyshev bound may be used as an indicator for the sub-optimality and hence the quality of the mean-variance approximation to the VaR constrained problem.

We then provide a method of replicating the optimal payoff function with a risk free bond, a forward contract and a spectrum of call and put options with different strike prices (Section 5). The portfolio is designed to meet a value at risk (VaR) constraint on the net hedged revenue of an entity holding a fixed price load following obligation. The results are illustrated through a numerical example (Section 6).

2 VaR-constrained Hedging Problem

We define VaR as a maximum possible loss at a $(1-\gamma)$ confidence level. In other words, VaR is the $(1-\gamma)$ percentile of the loss distribution.[2] In this section, we present a model for the hedging portfolio subject to a VaR limit set by the risk manager for a specified horizon. This preset VaR level will reflect the risk tolerance of the risk manager.

Consider the LSE whose revenue is determined by a fixed retail price r and the uncertain demand q. Denoting uncertain wholesale electricity price per unit as p, the profit $y(p,q)$ from retail sales at time 1 depends on the two random variable p and q. I.e.,

$$y(p,q) = (r-p)q.$$

Let LSE's beliefs on the realization of spot price p and load q be characterized by a joint probability function $f(p,q)$ for positive p and q, which is defined on the probability measure P.

Suppose the LSE hedges the profit through an exotic electricity option maturing at time 1. Let $Y(x)$ be the hedged profit, then

$$Y(x) = y(p,q) + x(p) = (r-p)q + x(p)$$

where $x(p)$ is a payoff function of the exotic option, which is contingent on the price of p.

[2] Loss is negative profit.

With the VaR limit V_0, the VaR-constrained hedging problem is formulated as follows:

$$\begin{array}{cl} \max_{x(p)} & E[Y(x)] \\ s.t. & E^Q[x(p)] = 0 \\ & VaR_\gamma(Y(x)) \leq V_0 \end{array} \quad (1)$$

where

$$VaR_\gamma(X) = \nu \text{ such that } P\{X \geq -\nu\} = 1 - \gamma.$$

for a random variable X, and with $E[\cdot]$ and $E^Q[\cdot]$ denoting expectations under the probability measures P and Q, respectively. The formulation seeks the payoff function of a self-financing hedging portfolio at time 1, which maximizes the expected profit while requiring that a $1-\gamma$ percentile of the loss distribution does not exceed V_0.[3]

The zero-cost constraint $E^Q[x(p)] = 0$ [4] requires the manufacturing cost[5] of the portfolio to be zero under a constant risk-free rate. This zero-cost constraint implies that purchasing derivative contracts may be financed from selling other derivative contracts or through money market accounts. In other words, under the assumption that there is no limit on the possible amount of instruments to be purchased and money to be borrowed, the model finds a portfolio from which the LSE obtains the maximum expected utility over total profit.

One might question the use of the optimal payoff function solved from the formulation (1). The optimal payoff function will eventually be used to derive the optimal quantities of forwards and options at different strike prices of which the hedging portfolio consists. This approach of getting the payoff

[3]The another possible formulation is:

$$\begin{array}{cl} \min_{x(p)} & VaR_\gamma(Y(x)) \\ s.t. & E^Q[x(p)] = 0 \\ & E[Y(x)] \geq \mu \end{array} \quad (2)$$

But one can prove that this formulation can be solved the same way as the above formulation (1)

[4]Q is a risk-neutral probability measure under which the hedging instruments are priced. Because the electricity market is incomplete, there may exist infinitely many risk-neutral probability measures. In this paper, it is assumed that a specific measure, Q, was picked according to some criteria. There are many proposed criteria to choose the risk-neutral measure in incomplete markets. See Xu [Xu04] for this subject.

[5]A derivative price is an expected value of the discounted payoff under the risk-neutral measure Q. We ignore here transaction costs.

function first and then calculating the portfolio composition that replicates the payoff, not only makes the problem solvable but also provides valuable insights regarding the optimal hedging portfolio.

3 Optimal Payoff Function in the Mean-Variance Efficient Frontier

The VaR constraint in the formulation (1) cannot be written in a tractable form for optimization without very restrictive assumptions on the distribution of $Y(x)$. If $Y(x)$ is linear in the risk factors which are normally distributed, then it is possible to write VaR in a closed form. However, in the formulation (1), $Y(x)$ has a multiplicative term of two risk factors and, moreover, a term of the unknown function $x(p)$. Thus, a closed form of $VaR(Y(x))$ cannot be obtained in a form amenable to simple optimization.

The reason behind the normal distribution having been a common assumption when calculating VaR is the fact that the quantiles of the normal distribution (actually, VaR) can be expressed using mean and variance. Likewise, when VaR can be expressed using mean and variance - even in cases when a closed form of the VaR cannot be obtained - the VaR-constrained problem could be solved using the mean-variance framework.

Therefore, a key assumption throughout this section is that $VaR(Y(x))$ is solely determined by mean and variance of $Y(x)$. In the following theorem adopted from Kleindorfer and Li [KL05] we show that under such an assumption, monotonicity of the VaR in the mean and variance of the $Y(x)$ corresponding to feasible hedging functions $x(p)$ is sufficient to ensure that the mean-maximizing VaR-constrained solution to (1) lies on the efficient mean-variance frontier.

Theorem 1 *Let*

$$X(p) = \{x(p) : x(p) \text{ is a continuous function of } p \text{ such that } E^Q[x(p)] = 0\}.$$
$$\Psi = \{Y(x) : Y(x) = y(p,q) + x(p) \text{ where } x(p) \in X(p)\},$$
$$E = \{E[Y(x)] : Y(x) \in \Psi\}, \;\; \Sigma = \{\sigma(Y(x)) : Y(x) \in \Psi\},$$

and $\Re$ be a set of real numbers. Let's define $VaR_\gamma(Y(x))$ as ν such that

$$P\{Y(x) \geq -\nu\} = 1 - \gamma.$$

Suppose now that there exists a continuous function $h : (E, \Sigma, \gamma) \to \Re$ that satisfies

$$VaR_\gamma(Y(x)) = h(\mu, \sigma, \gamma)$$

with $h(\mu, \sigma, \gamma)$ which is increasing in σ and non-increasing in μ for $\mu = E[Y(x)]$ and $\sigma^2 = V(Y(x))$.[6].

Then if $x^*(p)$ solves the problem (1), then the following (a) $\sim$ (e) hold:

(a) $x^*(p)$ is on the efficient frontier of the (E-VaR$_\gamma$) plane[7], on which any feasible $x(p)$ is mapped to a corresponding point $(VaR_\gamma(Y(x)), E[Y(x)])$.

(b) $x^*(p)$ is on the efficient frontier of the (E-V) plane[8], on which any feasible $x(p)$ is mapped to a corresponding point $(V(Y(x)), E[Y(x)])$.

(c) The variance on the efficient frontier in the (E-V) plane is non-decreasing in the mean.

(d) The variance on the efficient frontier in the (E-V) plane is a convex function of the mean.

(c) There exists $k > 0$ such that $x^*(p)$ solves

$$\max_{x(p) \in X(p)} E[Y(x)] - \frac{1}{2} k V(Y(x)). \tag{3}$$

Proof.

(a): Obvious.

(b): From (a), $x^*(p)$, the optimal solution to (1), is on the efficient frontier of (E-VaR$_\gamma$) plane. Now, consider some alternative $x(p) \in X(p)$ that can reduce the variance without reducing the mean of the $Y(x)$ distribution.

[6] $V(\cdot)$ denotes variance

[7] Efficient frontier of the (E-VaR$_\gamma$) plane is a set of points $(VaR_\gamma(Y(x)), E[Y(x)])$ in $(E - VaR_\gamma)$ plane for any feasible x such that $VaR_\gamma(Y(x')) \geq VaR_\gamma(Y(x))$, for any feasible x' with $E[Y(x')] \geq E[Y(x)]$

[8] Efficient frontier of (E-V) plane is a set of points $(V(Y(x)), E[Y(x)])$ in (E-V) plane for any feasible x such that $V(Y(x')) \geq V(Y(x))$ for any feasible x' with $E[Y(x')] \geq E[Y(x)]$

i.e., $\mu \geq \mu^*$ where $\mu = E[Y(x)]$ and $\mu^* = E[Y(x^*)]$ and $\sigma^2 < \sigma^{*2}$ where $\sigma^2 = V(Y(x))$, and $\sigma^{*2} = V(Y(x^*))$. Then, since h is non-increasing in μ and increasing in σ,

$$VaR_\gamma(Y(x)) = h(\mu, \sigma, \gamma) \leq h(\mu^*, \sigma, \gamma) < h(\mu^*, \sigma^*, \gamma) = VaR_\gamma(Y(x^*)).$$

However, this contradicts the assumption that $x^*(p)$ is on the efficient frontier in the $(E - VaR_\gamma)$ plane. This implies that for a fixed γ a feasible perturbation on $x^*(p)$ that solves (1) cannot reduce the variance of the $Y(x)$ distribution without increasing the mean. Hence $x^*(p)$ is also on the efficient frontier in the $(E - V)$ plane.

(c): Obvious from the definition of the efficient frontier.

(d): Consider (σ_1^2, μ_1) and (σ_2^2, μ_2) on the efficient mean-variance frontier corresponding to feasible hedging function $x_1(p)$ and $x_2(p)$ respectively. Without loss of generality assume that $\mu_2 > \mu_1$ and by monotonicity of the mean-variance frontier $\sigma_2 > \sigma_1$.

Now consider $x_3(p) = \alpha x_1(p) + (1-\alpha)x_2(p)$ for some $\alpha \in [0, 1]$ and denote $\mu_3 \equiv E[Y(x_3)] = \alpha\mu_1 + (1-\alpha)\mu_2$ and $\sigma_3^2 \equiv V(Y(x_3))$. Clearly by linearity of the mean,

$$\mu_3 = \alpha\mu_1 + (1-\alpha)\mu_2,$$

and $x_3(p) \in X(p)$. It follows from

$$\sigma_3^2 = \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 + 2\alpha(1-\alpha)\sigma_1\sigma_2\rho$$

where $\rho \equiv Corr(Y(x_1), Y(x_2))$ that

$$\begin{aligned}\sigma_3^2 - (\alpha\sigma_1^2 + (1-\alpha)\sigma_2^2) &= -\alpha(1-\alpha)\sigma_1^2 - \alpha(1-\alpha)\sigma_2^2 + 2\alpha(1-\alpha)\sigma_1\sigma_2\rho \\ &= -\alpha(1-\alpha)(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho) \\ &= -\alpha(1-\alpha)V(Y(x_1 - x_2)) \leq 0,\end{aligned}$$

because $\alpha \in [0, 1]$ and the variance is always nonnegative. Therefore, $\sigma_3^2 \leq \alpha\sigma_1^2 + (1-\alpha)\sigma_2^2$, which proves concavity of the variance as function of the mean on the efficient frontier if (σ_3^2, μ_3) is also on the efficient frontier.

To prove that (σ_3^2, μ_3) is on the mean-variance (M-V) efficient frontier, it is sufficient to show that any feasible solution which yields mean larger than μ_3 has larger variance than σ_3^2.

Now, let's consider a feasible $x(p)$ with $E[Y(x)] > \mu_3$. The proof is done if we show $V(Y(x)) > V(Y(x_3))$.

Note that $\hat{x} = x(p) - x_3(p)$ is also a feasible solution with positive mean. Since $x_3 = \alpha x_1 + (1-\alpha)x_2$ and $x = x_3 + \hat{x}$, we have

$$Y(x) = \alpha Y(x_1) + (1-\alpha)Y(x_2) + \hat{x} = \alpha Y(x_1 + \hat{x}) + (1-\alpha)Y(x_2 + \hat{x})$$

The last equality holds because $Y(x) \equiv Y(x(p)) = (r-p)q + x(p)$

Because x_1 and x_2 is in the efficient frontier and $E[Y(x_i+\hat{x})] > E[Y(x_i)]$, we have, for $i = 1, 2$,

$$V(Y(x_i + \hat{x})) > V(Y(x_i)). \tag{4}$$

Because $V(Y(x_i+\hat{x})) = V(Y(x_i)+\hat{x}) = V(Y(x_i)) + V(\hat{x}) + 2Cov(Y(x_i), \hat{x})$, it follows from Eq (4) that

$$2Cov(Y(x_i), \hat{x}) > -V(\hat{x}). \tag{5}$$

This leads to the following inequality:

$$\begin{aligned}
&Cov(Y(x_1 + \hat{x}), Y(x_2 + \hat{x})) = Cov(Y(x_1) + \hat{x}, Y(x_2) + \hat{x}) \\
&= Cov(Y(x_1), Y(x_2)) + V(\hat{x}) + Cov(Y(x_1), \hat{x}) + Cov(Y(x_2), \hat{x}) \\
&> Cov(Y(x_1), Y(x_2)) + V(\hat{x}) - \tfrac{1}{2}V(\hat{x}) - \tfrac{1}{2}V(\hat{x}) \\
&= Cov(Y(x_1), Y(x_2))
\end{aligned}$$

Now we have $V(Y(x)) > V(Y(x_3))$ because

$$\begin{aligned}
&V(Y(x)) \\
&= \alpha^2 V(Y(x_1 + \hat{x})) + (1-\alpha)^2 V(Y(x_2 + \hat{x})) + 2\alpha(1-\alpha)Cov(Y(x_1 + \hat{x}), Y(x_2 + \hat{x})) \\
&> \alpha^2 V(Y(x_1)) + (1-\alpha)^2 V(Y(x_2)) + 2\alpha(1-\alpha)Cov(Y(x_1), Y(x_2)) \\
&= V(\alpha Y(x_1) + (1-\alpha)Y(x_2)) = V(Y(x_3))
\end{aligned}$$

(The inequality comes from Eq. (4) and (5).)

(e) The concavity in conjunction with the non-decreasing property of the efficient mean-variance frontier implies that for any $x(p)$ on that frontier there exists a unique $k > 0$ such that $x(p)$ solves (3). In particular, this applies to $x^*(p)$.

Theorem 2 *Let*

$$x^k(p) = \arg \max_{x(p) \in X(p)} E[Y(x)] - \frac{1}{2}kV(Y(x)).$$

Then $E[Y(x^k)]$ and $V(Y(x^k))$ are monotonically non-increasing in k.

Proof.

Let $k_2 > k_1 > 0$ and denote for simplicity $Y(x^{k_i}) = Y_i$ for $i = 1, 2$. Then

$$E(Y_1) - k_1 V(Y_1) \geq E[Y_2] - k_1 V(Y_2)$$

$$E(Y_2) - k_2 V(Y_2) \geq E[Y_1] - k_2 V(Y_1)$$

Adding the two inequalities gives

$$(k_2 - k_1)V(Y_1) \geq (k_2 - k_1)V(Y_2)$$

implying $V(Y_1) \geq V(Y_2)$. Also we have from the first inequality,

$$E(Y_1) - E[Y_2] \geq k_1(V(Y_1) - V(Y_2)) \geq 0$$

So we have

$$E[Y_1] \geq E[Y_2]$$

QED.

Theorem 1 states that the feasible set of the VaR-constrained problem is restricted to the solutions of mean-variance problems for varying k. Therefore, the solution to (1) can be obtained in the following way:

(a) Obtain

$$x^k(p) = \arg \max_{x(p) \in X(p)} E[Y(x)] - \frac{1}{2} k V(Y(x)).$$

(b) For each k, calculate $VaR(k) \equiv VaR(Y(x^k))$ such that

$$P\{Y(x^k) \geq -VaR(k)\} = 1 - \gamma$$

(c) Find k such that $VaR(k) \leq V_0$ that maximizes $E[x^k(p)]$. i.e.,

$$x^*(p) = x^{k^*}(p), \text{ where } k^* = \arg \max_k E[x^k(p)] \text{ s.t. } VaR_\gamma(k) \leq V_0$$

By Theorem 2, such k is the smallest k with $VaR_\gamma(k) \leq V_0$.

4 The Optimal Payoff Function when the Demand and Log Price Follows Bivariate Normal Distribution

It is often assumed that the electricity demand and logarithm of price are normally distributed with some correlation. In Proposition 1 and Lemma 1, we show that under such assumption a closed form of $x^k(p)$ can be obtained. We have also shown in the previous section that $E[Y(x^k(p))]$ and the variance $V[Y(x^k(p))]$ are non-increasing in k. We will now describe an approximation procedure that searches for an approximate solution to the VaR-constrained Expected-value-maximizing self-financed hedging function along the mean-variance efficient frontier. The justification for this approximation is motivated by the intuitively plausible properties of the VaR that make such an approximation exact. The approximation is also supported by the fact that the required properties are met by the Chebyshev upper bound[9] on the VaR so that tightening the VaR constraints by replacing the VaR with its Chebyshev approximation will also produce results that lie on the mean-variance efficient frontier.

To obtain the approximate solution we characterized above, we start with $k = \epsilon$. (ϵ is a small constant). Using the formula for $x^k(p)$ given in (8), we compute the corresponding $VaR_\gamma(k) \equiv VaR_\gamma(Y(x^k(p)))$ using a Monte-Carlo simulation such that

$$P\{(r-p)q + x^k(p) \geq -VaR_\gamma(k)\} = 1-\gamma.$$

We then repeat the process incrementing k until $VaR_\gamma(k) \leq V_0$ at which point we set $k^* = k$. The monotonicity of the mean in k coming from Theorem 2 guarantees that the first k at which the VaR constraint is satisfied will yield the largest expected value.

[9]Chebyshev's inequality says that $P\{|X - E[X]| \geq k\} \leq \sigma^2(X)/k^2$ for any random variable X with finite mean $E[X]$ and variance $\sigma^2(X)$, and any positive real number t. With $k = t\sigma(X)$, $P\{|X - E[X]| \geq t\,\sigma(X)\} \leq \frac{1}{t^2}$. It follows that

$$P\{X \leq E[X] - t\,\sigma(X)\} \leq \frac{1}{t^2}$$

Therefore, $VaR_{1-\frac{1}{t^2}}(X) \leq t\,\sigma(X) - E[X]$. [AB02]

Proposition 1 *Maximizing the mean-variance utility function on profit,*

$$E[U(Y)] = E[Y] - \frac{1}{2}kVar_{\gamma}(Y),$$

yield an optimal solution $x^*(p)$ *to problem*

$$\begin{aligned} \max_{x(p)} \ & E\big[U[Y(p,q,x(p))]\big] \\ s.t. \ & E^Q[x(p)] = 0 \end{aligned} \tag{6}$$

that is given by

$$x^*(p) = \frac{1}{k}\Big(1 - \frac{\frac{g(p)}{f_p(p)}}{E^Q[\frac{g(p)}{f_p(p)}]}\Big) - E[y(p,q)|p] + E^Q[E[y(p,q)|p]]\frac{\frac{g(p)}{f_p(p)}}{E^Q[\frac{g(p)}{f_p(p)}]} \tag{7}$$

where $f_p(p)$ *is the marginal distribution of* p *under probability measure* P*, and* $g(p)$ *is the probability density function of* p *under risk-neutral measure* Q*.*

Proof The proof is given in the Appendix.

Lemma 1 *Suppose the marginal distributions of* p *and* q *are as follows:*

$$\begin{aligned} &\textit{Under } P: \quad \log p \sim N(m_1, s^2), \quad q \sim N(m, u^2), \quad Corr(\log p, q) = \rho \\ &\textit{Under } Q: \quad \log p \sim N(m_2, s^2) \end{aligned}$$

Then, the solution to (3) is

$$x^k(p) = \frac{1}{2k}(1 - Ap^B) - (r-p)\big(m + \rho\frac{u}{s}(\log p - m_1)\big) + CAp^B \tag{8}$$

where the constants are

$$\begin{aligned} A \ &= e^{-\frac{(m_1-m_2)(m_1-3m_2)}{2s^2}}, \quad B = \tfrac{m_2-m_1}{s^2} \\ C \ &= (r - e^{m_2+\frac{1}{2}s^2})(m - \rho\tfrac{u}{s}m_1) + \rho\tfrac{u}{s}(rm_2 - (m_2+s^2)e^{m_2+\frac{1}{2}s^2}). \end{aligned}$$

Proof. From proposition 1, the optimal solution of (3) is:

$$x^k(p) = \frac{1}{2k}\Big(1 - \frac{\frac{g(p)}{f_p(p)}}{E^Q[\frac{g(p)}{f_p(p)}]}\Big) - E[y(p,q)|p] + E^Q[E[y(p,q)|p]]\frac{\frac{g(p)}{f_p(p)}}{E^Q[\frac{g(p)}{f_p(p)}]} \tag{9}$$

From a density function of lognormal distribution, we have

$$\frac{g(p)}{f_p(p)} = \frac{\frac{1}{ps\sqrt{s\pi}}\exp\big(-\frac{1}{2}\big(\frac{\log p - m_2}{s}\big)^2\big)}{\frac{1}{ps\sqrt{s\pi}}\exp\big(-\frac{1}{2}\big(\frac{\log p - m_1}{s}\big)^2\big)} = \exp\left(\frac{m_2 - m_1}{s^2}\log p + \frac{m_1^2 - m_2^2}{2s^2}\right).$$

Since $\frac{m_2-m_1}{s^2}\log p + \frac{m_1^2-m_2^2}{2s^2} \sim N\big(\frac{m_2-m_1}{s^2}m_2 + \frac{m_1^2-m_2^2}{2s^2}, (\frac{m_2-m_1}{s^2})^2 s^2\big)$ under Q, we obtain $E^Q\big[\frac{g(p)}{f_p(p)}\big] = \exp\big(\frac{m_2-m_1}{s^2}m_2 + \frac{m_1^2-m_2^2}{2s^2} + \frac{1}{2}(\frac{m_2-m_1}{s^2})^2 s^2)\big) = \exp\big(\frac{(m_1-m_2)^2}{s^2}\big)$ and thus,

$$\begin{aligned}\frac{\frac{g(p)}{f_p(p)}}{E^Q\big[\frac{g(p)}{f_p(p)}\big]} &= \exp\big(\frac{m_2-m_1}{s^2}\log p + \frac{m_1^2-m_2^2}{2s^2} - \frac{(m_1-m_2)^2}{s^2}\big) \\ &= e^{-\frac{(m_1-m_2)(m_1-3m_2)}{2s^2}} p^{\frac{m_2-m_1}{s^2}}\end{aligned} \tag{10}$$

On the other hand,

$$E[y(p,q)|p] = E[(r-p)q|p] = (r-p)E[q|p] = (r-p)\big(m + \rho\frac{u}{s}(\log p - m_1)\big), \tag{11}$$

and thus,

$$\begin{aligned}E^Q[y(p,q)|p] &= (r - E^Q[p])(m - \rho\tfrac{u}{s}m_1) + \rho\tfrac{u}{s}(rE^Q[\log p] - E^Q[p\log p]) \\ &= (r - e^{m_2+\frac{1}{2}s^2})(m - \rho\tfrac{u}{s}m_1) + \rho\tfrac{u}{s}(rm_2 - (m_2+s^2)e^{m_2+\frac{1}{2}s^2})\end{aligned} \tag{12}$$

Plugging (10),(11), and (12) into (9) results in Eq. (8). **QED.**

5 Replication of Exotic Payoffs

Once the optimal payoff function is obtained by the algorithm given in the previous section, we construct a portfolio composed of standard instruments that replicates the exotic payoff function obtained.

Carr and Madan [CM01] showed that any twice continuously differentiable function $x(p)$ can be written in the following form:

$$x(p) = [x(s) - x'(s)s] + x'(s)p + \int_0^s x''(K)(K-p)^+ dK + \int_s^\infty x''(K)(p-K)^+ dK$$

for an arbitrary positive s.[10] This formula suggests a way of replicating the

[10] The simplest way of proving the formula is as follows: $\int_0^s x''(K)(K-p)^+dK + \int_s^\infty x''(K)(p-K)^+dK = \int_s^p x''(K)(p-K)dK = [x'(K)(p-K)]_s^p + \int_s^p x'(K)dK = -x'(s)(p-s) + x(p) - x(s)$; the first equality was obtained by considering the both cases of $p < s$ and $p \geq s$, and the second equality results from the integration by part.

payoff function $x(p)$. Let F be the forward price for delivery at time 1. Evaluating the equation at $s = F$ and rearranging it gives

$$x(p) = x(F) \cdot 1 + x'(F)(p - F) + \int_0^F x''(K)(K - p)^+ dK + \int_F^\infty x''(K)(p - K)^+ dK. \quad (13)$$

Note that $1, (p - F), (K - p)^+$ and $(p - K)^+$ in the above expression represent payoffs at time 1 of a bond, forward contract, European put options, and European call options, respectively. Therefore, an exact replication can be obtained from a long cash position of size $x(F)$, a long forward position of size $x'(F)$, long positions of size $x''(K)dK$ in puts struck at K, for a continuum of $K < F$, and long positions of size $x''(K)dK$ in calls struck at K, for a continuum of $K > F$. Note that unless the optimal payoff function is linear, the optimal strategy involves purchasing (or selling short) a spectrum of both call and put options with continuum of strike prices. This result demonstrates that in order to hedge price and quantity risks together, LSEs should purchase a portfolio of options. The strike prices of call options effectively work as price caps on load increments. In practice, electricity derivatives markets, as any derivatives markets, are incomplete. Consequently, the market does not offer options for the full continuum of strike prices, but typically only a small number of strike prices are offered. To implement the above replicating strategy using a discrete set of standard options contracts, we need to discretize the strike prices and approximate the optimal payoff function using a set of discrete option at the available strike prices. We provide here an approximate replication of an exotic payoff function using the existing Vanilla options so that the total payoff from those options is close to the exotic payoff. Suppose there are put options with strike prices $K_1 < \cdots < K_n = F$ and call options with strike prices $F = K'_1 < \cdots < K'_m$ in the market. Letting $K_{n+1} = K_n$, $K_0 = 0$, $K'_0 = K'_1$, and $K'_{m+1} = \infty$, consider the following strategy, which consists of

a long cash position of size $x(F)$,
a long forward position of size $x'(F)$,
long positions of size $\frac{1}{2}(x'(K_{i+1}) - x'(K_{i-1}))$ in puts struck at K_i, $(i = 1, \cdots, n)$,
long positions of size $\frac{1}{2}(x'(K'_{i+1}) - x'(K'_{i-1}))$ in calls struck at K'_i $(i = 1, \cdots, m)$.

This strategy was obtained by the following approximations: $\int_0^F x''(K)(K-p)^+dK + \int_F^\infty x''(K)(p-K)^+dK$
$= \sum_{i=0}^{n-1} \int_{K_i}^{K_{i+1}} x''(K)(K-p)^+dK + \sum_{i=1}^{m} \int_{K'_i}^{K'_{i+1}} x''(K)(p-K)^+dK$

$$\approx \sum_{i=0}^{n-1} \int_{\max(p,K_i)}^{\max(p,K_{i+1})} x''(K)dK \cdot \frac{1}{2}\{(K_i-p)^+ + (K_{i+1}-p)^+\}$$
$$+ \sum_{i=1}^{m} \int_{\min(p,K'_i)}^{\min(p,K'_{i+1})} x''(K)dK \cdot \frac{1}{2}\{(p-K'_i)^+ + (p-K'_{i+1})^+\}$$

$$\approx \sum_{i=0}^{n-1} \int_{K_i}^{K_{i+1}} x''(K)dK \cdot \frac{1}{2}\{(K_i-p)^+ + (K_{i+1}-p)^+\}$$
$$+ \sum_{i=1}^{m} \int_{K'_i}^{K'_{i+1}} x''(K)dK \cdot \frac{1}{2}\{(p-K'_i)^+ + (p-K'_{i+1})^+\}$$

$= \sum_{i=1}^{n} \int_{K_{i-1}}^{K_{i+1}} x''(K)dK \cdot \frac{1}{2}(K_i-p)^+ + \sum_{i=1}^{m} \int_{K'_{i-1}}^{K'_{i+1}} x''(K)dK \cdot \frac{1}{2}(p-K'_i)^+.$

In this approximation scheme, the error will be small if $x''(p)$ is a constant in each interval between two consecutive strike prices, and when price realizations p are close to the discrete strike prices. The error can be reduced by refining the strike price discretization in the range were there is a high probability that p will fall.

6 An Example

In this section we demonstrate the computation of an approximate optimal VaR-constrained volumetric hedging problem using the method developed in the previous section. Consider a hypothetical LSE that charges a flat retail rate $r = \$120/MWh$ to its customers. The wholesale spot price p at which the LSE must purchase its power and and the load q it is obligated to serve in any fixed time interval (typically 15 minutes), are distributed according to a bivariate distribution in quantity and log price:

Under P: $\log p \sim N(4, 0.7^2)$, $\quad q \sim N(3000, 600^2)$, $\quad Corr(\log p, q) = 0.8$
Under Q: $\log p \sim N(4.1, 0.7^2)$

Note that we assume here $P \neq Q$. Otherwise, the mean-variance problem has the same solution for all k, In such case, the VaR-constrained problem either has the same solution as the variance-minimizing problem, or is infeasible.

Figure 1 shows a distribution of unhedged profit,

$$y(p, q) = (120 - p)q.$$

95% VaR is also indicated in the figure, which is about \$20,000. The mean of the distribution is \$127,000. This implies that there is 5% chance that the LSE can take a loss of more than \$20,000.

The VaR-constrained problem for the LSE which seeks a hedging strategy that maximizes the expected profit with at least \$60,000 profit with 95% probability is formulated as follows:

$$\begin{array}{cl} \max_{x(p)} & E[Y(x)] \\ s.t. & E^{Q}[x(p)] = 0 \\ & VaR_{\gamma}(Y(x)) \leq -60000 \end{array} \tag{14}$$

where $Y(x) = (120 - p)q + x(p)$ and $Pr\{Y(x) \geq -VaR_{\gamma}(Y(x))\} = 0.95$.

Motivated by Theorem 1 we restrict our search for solution to the VaR constrained problem to optimal solutions for the mean-variance problems for various risk-aversion levels k and for each such candidate solution we compute the corresponding VaR. The relationship between VaR and k is drawn in Figure 2 as an example. The figure also shows the mean of the hedged profit, $E[Y(x^k)]$, on the right axis, which is non-increasing in k as proven in Theorem 2. Because of the monotonicity of the mean in k selecting the first value of k that meets the VaR constraint as $k^* = 3.5 \times 10^{-6}$ gives the largest mean value with $-VaR_{\gamma}(k) \geq 60000$ among all hedging portfolios that maximize a mean-variance criterion.

Figure 3 illustrates the mean-variance efficient frontier and the corresponding mean-VaR frontier for our example. Note that the mean-VaR frontier is the efficient mean-VaR frontier only if the distribution of hedged profit satisfies the monotonicity properties postulated in Theorem 1.

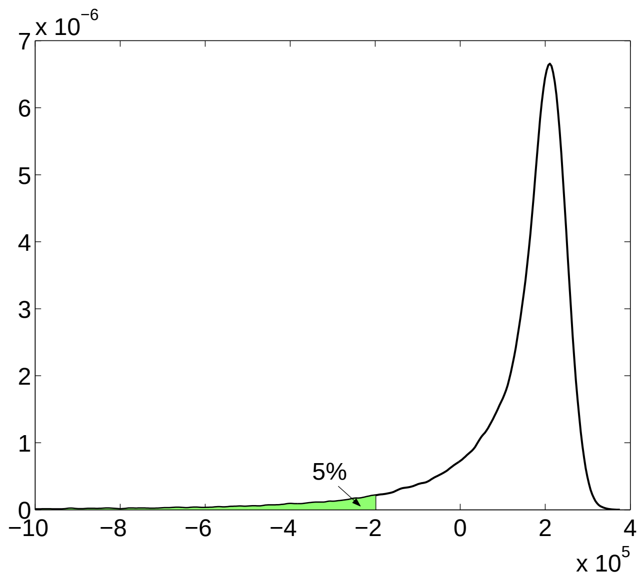


Figure 1: Distribution of the unhedged profit $y(p, q) = (r - p)q$

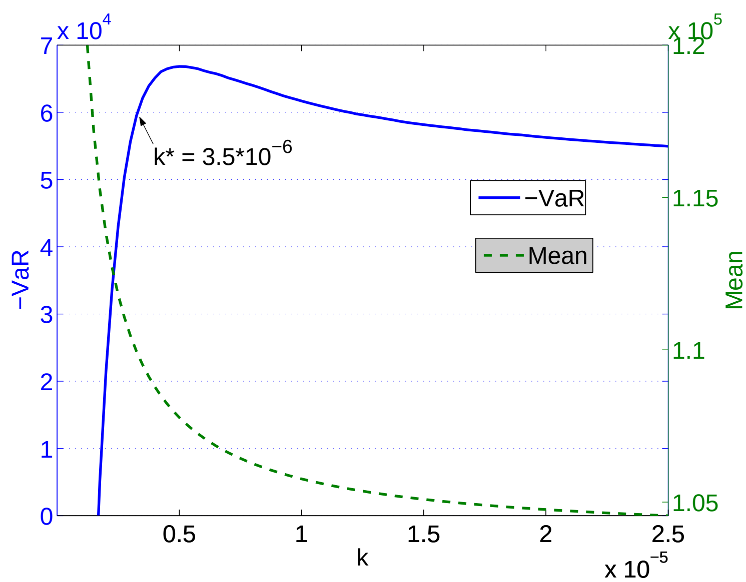
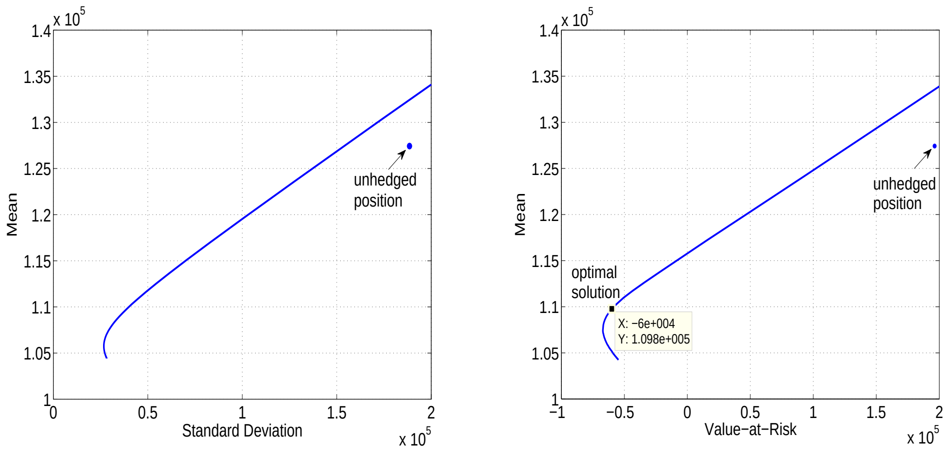
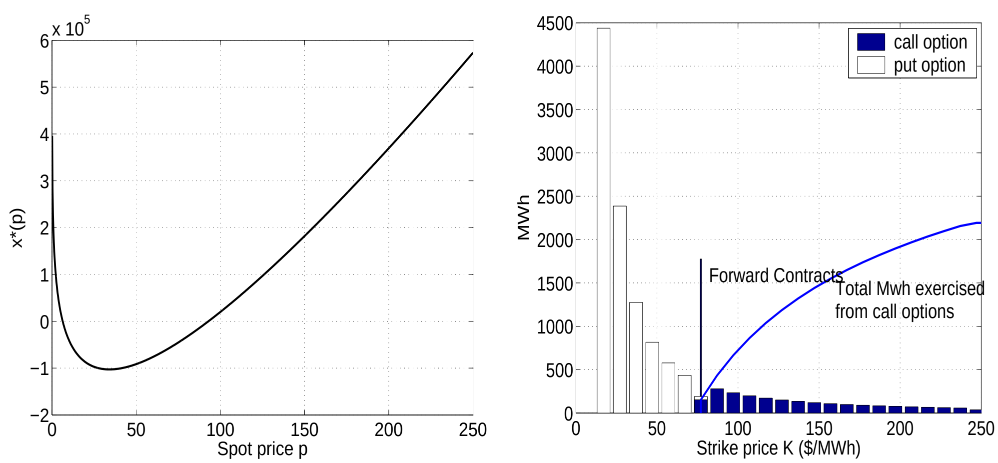


Figure 2: $-VaR(k)$ in the left y-axis and $E[Y(x^k(p))]$ in the right y-axis. The optimal k^* is obtained as the first k that provides -VaR no less than the required level $60,000$.



(a) Mean-variance frontier (b) Mean-VaR frontier

Figure 3: Mean-variance frontier and mean-VaR frontier



(a) The optimal payoff function (b) Replicating strategy

Figure 4: Hedging strategy for an LSE that maximizes the expected payoff with VaR constraints of $-\$60,000$. The underlying distributions of spot prices and load are $\log p \sim N(4, 0.7^2)$, $q \sim N(3000, 600^2)$, and $Corr(\log p, q) = 0.8$ (assuming $r = \$120/MWh$)

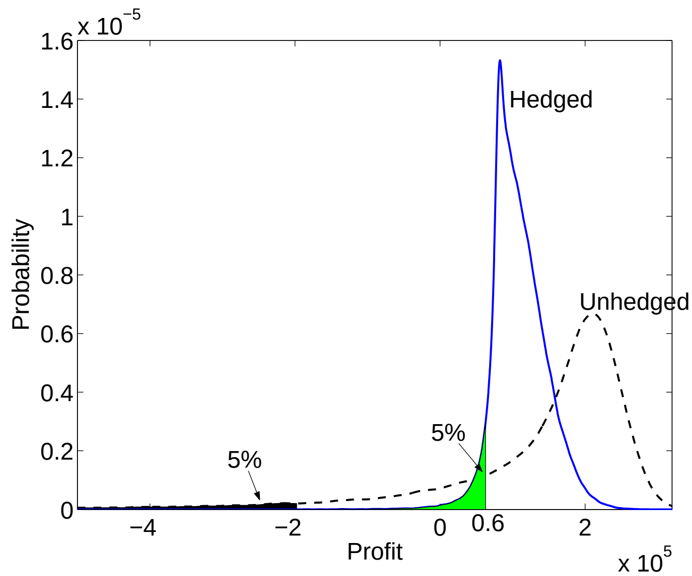


Figure 5: Profit distributions and VaRs before and after the optimal hedge

The optimal mean-variance hedging strategy corresponding to k^* and hence, the approximation to the optimal mean-VaR hedging strategy, is shown in Figure 4. Figure 4(a) shows the payoff function $x^*(p) \equiv x^{k^*}(p)$ obtained as an approximation for the VaR-constrained problem, and Figure 4(b) illustrates its replicating strategy consisting of forwards, calls, and puts, as described in Section 5.

Figure 5 compares profit distributions before and after hedging. One can see that the hedge obtained as an approximate solution to the VaR-constrained problem reduces the left-tail of the profit distribution significantly.

Figure 6 shows the profit distributions for different k. The corresponding -VaR is represented as the vertical line from the distribution to the x-axis. $k = 3.5 \times 10^{-6}$ corresponds to profit after the optimal hedge. One can see that $k = 2 \times 10^{-6}$ gives the higher expected value, 1.13×10^5, than the optimal one, but it was rejected from the feasible hedge because its VaR level exceeds the required level of $-\$60,000$. The graph for $k = 5 \times 10^{-6}$ shows a case of VaR satisfying the required level, but it was not chosen for the optimum

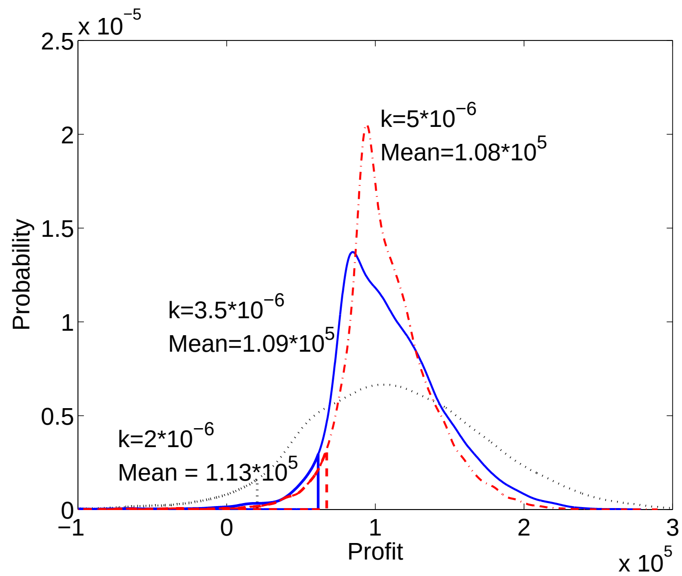


Figure 6: Profit distribution and its VaR for various levels of k

since it provides a lower expected profit than the optimal one.

6.1 Conclusion

This paper developed a method of mitigating price and volumetric risk that load-serving entities (LSEs) and marketers of default service contract face in providing their customers' load following service at fixed or regulated prices while purchasing electricity or facing an opportunity cost at volatile wholesale prices.

Exploiting the inherent positive correlation and multiplicative interaction between wholesale electricity spot price and demand volume, we developed a hedging strategy for the LSE's retail positions (which is in fact a short position on unknown volume of electricity) using electricity standard derivatives such as forwards, calls, and puts. The hedging strategy is intended to maximize the expected profit under the VaR constraint, which limits the lowest level below which the hedged profit wouldn't fall with 95% confidence.

However, VaR constrained problems are generally very hard to solve analytically unless the value or profit under consideration is normally distributed. In our case, the profit depends on the product of the two correlated variables. Moreover our hedging strategy is characterized by a nonlinear function of a random variable. We address this difficulty by limiting our search to feasible VaR-constrained self-financed hedging portfolios on the mean-variance efficient frontier. We provide theoretical justification to such an approximation and derive, an analytic representation of hedging portfolios on the mean-variance efficient frontier as function of the risk aversion factor.

The computation of an approximate solution to the VaR-constrained problem on the mean variance efficient frontier is facilitated by the fact that it corresponds to the smallest risk-aversion factor whose associated VaR meets the constraint limit.

When one uses the mean-variance formulation, it is usually easy to solve the problem, but hard to decide what the appropriate risk-aversion factor is. The analysis in this section implies that one can use a VaR-constrained formulation as an alternative, which takes one of the mean-variance solutions but automatically chooses associated risk aversion at which the maximum mean is achieved while maintaining the required VaR level. The advantage of using the VaR-constrained formulation is that VaR is easier to interpret, and it is a widely used risk-measure in practice.

To obtain a realistic hedging portfolio, we solved for the payoff function that represents the payoff of a costless exotic option as a function of price. We then showed how that exotic option can be replicated using a portfolio of forward contracts and European options.

While at present the liquidity of electricity options is limited, the use of call options has been advocated by Oren [Ore05] and Chao and Wilson[CW04] in the electricity market design literature as a tool for resource adequacy, market power mitigation, and spot volatility reduction. These authors advocated capacity payments in the form of option premiums that will incent capacity investment, and ensure electricity supply at a predetermined strike price. Better understanding of how call options can facilitate risk management associated with service obligations, capacity investment and energy trading will hopefully increase their use and liquidity.

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Appendix

Proof of Proposition 1

The Lagrangian function for the optimization problem (6) is given by

$$\begin{aligned} L(x(p)) &= E\big[U(Y(p,q,x(p)))\big] - \lambda E^Q[x(p)] \\ &= \int_{-\infty}^{\infty} E\big[U(Y)|p\big] f_p(p)dp - \lambda \int_{-\infty}^{\infty} x(p)g(p)dp \end{aligned}$$

with a Lagrange multiplier λ and the marginal density function $f_p(p)$ of p under P. Differentiating $L(x(p))$ with respect to $x(\cdot)$ results in

$$\frac{\partial L}{\partial x(p)} = E\Big[\frac{\partial Y}{\partial x} U'(Y)\Big|p\Big] f_p(p) - \lambda g(p) \tag{15}$$

by the Euler equation. Setting (15) to zero and substituting $\frac{\partial Y}{\partial x} = 1$ yields the first order condition for the optimal solution $x^*(p)$ as follows:

$$E\big[U'(Y(p,q,x^*(p)))\big|p\big] = \lambda^* \frac{g(p)}{f_p(p)} \tag{16}$$

Here, the value of λ^* should be the one that satisfies the constraint $E^Q[x(p)] = 0$

It follows from $Var(Y) = E[Y^2] - E[Y]^2$ that

$$U(Y) \equiv Y - \frac{1}{2}a(Y^2 - E[Y]^2).$$

From $U'(Y) = 1 - aY$, the optimal condition (16) is as follows:

$$E[1 - aY^*|p] = \lambda^* \frac{g(p)}{f_p(p)}.$$

Equivalently,

$$f_p(p) - aE[Y^*|p]f_p(p) = \lambda^* g(p). \tag{17}$$

Integrating both sides with respect to p from $-\infty$ to ∞, we obtain $\lambda^* = 1 - aE[Y^*]$. By substituting λ^* and $Y^* = y(p,q) + x^*(p)$ into (17) gives

$$f_p(p) - a\Big(E[y(p,q)|p] + x^*(p)\Big)f_p(p) = g(p) - a\Big(E[y(p,q)] + E[x^*(p)]\Big)g(p).$$

By rearranging, we obtain

$$x^*(p) = \frac{1}{a} - \frac{1}{a}\frac{g(p)}{f_p(p)} + \Big(E[y(p,q)] + E[x^*(p)]\Big)\frac{g(p)}{f_p(p)} - E[y(p,q)|p] \qquad (18)$$

To cancel out $E[x^*(p)]$ in the right-hand side, we take the expectation under Q to the both sides to obtain

$$0 = \frac{1}{a} - \frac{1}{a}E^Q\Big[\frac{g(p)}{f_p(p)}\Big] + \Big(E[y(p,q)] + E[x^*(p)]\Big)E^Q\Big[\frac{g(p)}{f_p(p)}\Big] - E^Q[E[y(p,q)|p]], \qquad (19)$$

and subtract Eq.(19)$\times\frac{g(p)/f_p(p)}{E^Q[g(p)/f_p(p)]}$ from Eq.(18). This gives the final formula for the optimal payoff function under mean-variance utility as

$$x^*(p) = \frac{1}{a}\Bigg(1 - \frac{\frac{g(p)}{f_p(p)}}{E^Q[\frac{g(p)}{f_p(p)}]}\Bigg) - E[y(p,q)|p] + E^Q[E[y(p,q)|p]]\frac{\frac{g(p)}{f_p(p)}}{E^Q[\frac{g(p)}{f_p(p)}]} \qquad (20)$$

QED.