

Optimal Static Hedging of Volumetric Risk in a Competitive Wholesale Electricity Market

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In competitive wholesale electricity markets, regulated load-serving entities (LSEs) and marketers with default service contracts have obligations to serve fluctuating load at predetermined fixed prices while meeting their obligation through combinations of long-term contracts, wholesale purchases, and self-generation that are subject to volatile prices or opportunity cost. Hence, their net profits are exposed to joint price and quantity risk, both of which are correlated with weather variations. In this paper, we develop a static hedging strategy for the LSE (or marketer) whose objective is to minimize a mean-variance utility function over net profit, subject to a self-financing constraint. Because quantity risk is nontraded, the hedge consists of a portfolio of price-based financial energy instruments, including a bond, forward contract, and a spectrum of European call and put options with various strike prices. The optimal hedging strategy is jointly optimized with respect to contracting time and the portfolio mix, which varies with contract timing, under specific price and quantity dynamics and the assumption that the hedging portfolio, which matures at the time of physical energy delivery, is purchased at a single point in time. Explicit analytical results are derived for the special case where price and quantity follow correlated Ito processes.

Key words: energy risk; competitive electricity markets; volumetric hedging; incomplete markets

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1. Introduction

Electricity is one of the most (if not the most) volatile traded commodities.¹ Numerous instances have been cited in the literature (see, e.g., Oum and Oren 2009) when electricity prices in the United States and abroad that normally range between \$30 to \$60 per megawatt hour rose for short durations to \$7,000 and \$10,000 per megawatt hour, and in some cases persisted for several days at \$1,000 per megawatt hour.² In most systems in the United States, prices are now capped at \$1,000 per megawatt hour, but persistent wholesale prices at that level can still have

severe economic consequences. In California during the 2000/2001 energy crisis, persistent electricity prices around \$500 per megawatt hour had devastating effects on the economy. These dire consequences were largely attributed to the fact that the major utilities, who were forced to sell power to their customers at low fixed prices set by the regulator, were not properly hedged through long-term supply contracts. Such bad experiences led regulators and market participants to recognize the importance and necessity of risk management in competitive electricity markets.

Load-serving entities (LSEs) providing regulated electricity service to a majority of retail customers in the United States and abroad are typically obligated to guarantee fixed electricity prices over extended time periods (subject to periodic regulatory review), although their customers are free to control their consumption with a flip of a switch. Furthermore, LSEs are uncertain about how much electricity their

¹ Typical volatilities include dollar/yen exchange rates (10%–20%), LIBOR rates (10%–20%), the S&P 500 index (20%–30%), the NASDAQ (30%–50%), natural gas prices (50%–100%), and spot electricity (100%–500% and higher) (see Eydeland and Wolyniec 2003).

² That happened in Texas during an ice storm that lasted three days in February 2004.

customers will use at a certain hour until the customers actually turn switches on and draw electricity. Hence, in addition to the wholesale price volatility faced by LSEs, they are also exposed to quantity uncertainty, often referred to as volumetric risk. Uncertainty or unpredictability of demand is a traditional concern for any commodity, but holding inventory is a good solution to mitigate quantity risk for those commodities that can be economically stored. Unfortunately, electricity once produced cannot be practically stored in large quantities. This is the most important characteristic that differentiates the electricity market from the money market or other commodities markets. Because electricity needs to be generated at the same time it is consumed, the traditional method of purchasing an excess quantity of a product when prices are low and holding inventories cannot be used by firms retailing electricity. Moreover, unlike other commodities, LSEs, which are typically regulated, operate under an obligation to serve and cannot curtail service to their customers (except under special service agreements) nor pass high wholesale prices on to their customers by charging more when they cannot procure electricity at favorable prices.³ Unlike telecom or Internet services, for instance, a busy signal or dropping of packets is intolerable. The common reliability standard for electricity service is no more than one day of involuntary curtailment in 10 years.

The exposure of the LSE to price and volumetric risk is amplified by the fact that both quantities and wholesale spot prices are largely driven by weather conditions, and hence they are strongly correlated. Typically, LSEs attempt to cover their service obligation and hedge their exposure to price risk by purchasing fixed-price forward contracts for the load level they expect to serve. Regardless of how they determine their forward cover, it is inevitable that the LSE is sometimes short and has to procure the balance on the wholesale spot market while at other times they have excess that they dispose of in the spot market. Because of the correlation between price and load level, the LSE is likely to be short and needs

to procure the balance on the wholesale spot market just when spot prices are high (and most likely above the regulated fixed retail price). Likewise, when the LSE needs to dispose of overcontracted amounts of electricity, spot prices are likely to be lower than its forward contract price. As a result of the adverse movement of prices and quantities, the LSE's profit is effectively doubly exposed to the price risk, and such exposure cannot be fully addressed with simple linear hedging strategies.

A similar problem is faced by marketers or generating companies that sold load-following fixed-price contracts in the default service auction in New Jersey. LSEs in New Jersey were ordered by the local public utility commissions to procure default service contracts from generators via auction (see Loxley and Salant 2004). The sellers of such contracts assume an obligation to provide a proportional slice of the fluctuating load at a fixed energy price set through the auction. Generators or marketers selling such contracts have their profit (if they procure in the wholesale market) or their opportunity cost (if they self-generate) exposed to price and volumetric risk.

In this paper, we develop a static hedging strategy that deals with price and volumetric risk. Specifically, we develop a hedging portfolio of standard financial instruments for electric power, such as forwards and European calls and puts, and co-optimize the portfolio mix and procurement time of the contracts. Although weather derivatives, whose underlying indices are strongly correlated with electricity demand, can also be an effective alternative in hedging volumetric risks, we do not include such instruments in our hedging portfolio. The speculative image of the weather derivatives makes them undesirable for a regulated utility having to justify to a regulator its risk management practices and the cost associated with such practices (which are passed on to customers). Moreover, weather derivatives cannot ensure supply adequacy, which is a major concern in the electricity industry. In some jurisdictions, the regulators (e.g., the California Public Utility Commission (CPUC)), who are motivated by concerns for generation adequacy, require that LSEs hedge their load-serving obligations and appropriate reserves with physically covered electricity forward contracts and

³ In fact, most of the U.S. states that opened their retail markets to competition have frozen their retail electricity prices.

options; that is, the hedges cannot be settled financially or subject to liquidation damages, but must be covered by specific installed or planned generation capacity capable of physical delivery. In California, the CPUC has explicitly ordered the phasing out of financial contracts with liquidation damages as a means of meeting generation adequacy requirements (see California Public Utility Commission 2004a, b). On the other hand, private marketers with load-following obligations may find weather derivatives attractive.

This paper extends work reported in two previous papers by the authors (Oum et al. 2006, Oum and Oren 2009) that address the joint price and quantity hedging problem faced by the holder of a fixed-price load-following obligation. In the first paper, Oum et al. (2006) developed a fixed single-period static hedging strategy where the hedging portfolio is contracted at the beginning of the period and exercised at the delivery time taking place at the end of the period. Specifically, Oum et al. (2006) obtained the optimal hedging strategy that uses electricity derivatives to hedge price and volumetric risks by maximizing the expected utility of the hedged profit. When such a portfolio is held by an LSE, the call options with strikes being below the spot price will be exercised so that the amount of the options being exercised is procured at the strike prices. Using this strategy, the LSE can set an increasing price limit on incremental load by paying the premiums for the options. This strategy is not only effective in managing quantity risk, but was also suggested in the market design literature such as Chao and Wilson (2004), Oren (2005), and Willems (2006) as a means to achieve resource adequacy, mitigate market power, and reduce spot price volatility. In Oum and Oren (2009), we solved the same problem as in Oum et al. (2006) but optimized the hedging portfolio so as to maximize expected profit subject to a value-at-risk (VaR) constraint. This paper extends the work reported by Oum et al. (2006) by relaxing the assumption that contracting is done at the onset of the period and by exploring optimal timing for procuring the hedging contracts. We will still assume, however, that the entire hedging portfolio is procured at a single point in time and co-optimize the mix and procurement time of the hedging portfolio using an expected utility criterion. Although we will

try to avoid duplication of previous work, some repetition may be unavoidable for the sake of coherence and for making this paper self-contained.

This paper is organized as follows: After a brief literature review in §2, we present in §3 the formulation and solution for the optimal payoff function of a static hedge given the procurement time, as well as the replicating portfolio that consists of forward, European call and put options, which yields the optimal payoff. This section reproduces the results of Oum et al. (2006), and hence many of the details will be omitted. Forwards and options prices that are included in our hedging portfolio change as the time approaches delivery time, reflecting the changing expectations in the market. Thus, the mix of the optimal hedging portfolio also changes with the hedging time. Our result shows that hedging too late can increase risk sharply. Optimizing such timing decisions, therefore, requires co-optimization of the hedging portfolio and contracting time. This problem is considered in §4, and the results are illustrated by means of a detailed example. Section 5 contains a summary and general conclusions.

2. Literature Review

In this section, we first review the literature addressing the problem of hedging nontraded risk in an incomplete market. Then we will discuss the limited literature that deals directly with the problem of hedging price and quantity risk in the context of electric power markets.

2.1. Hedging Nontraded Quantity Risk

Conventional hedging strategies typically deal with a single source of uncertainty, which is traded in the market through a variety of financial instruments whose payoffs are directly linked to the uncertain underlying quantity (e.g., commodity price). In many cases, however, more than one source of uncertainty interacts with each other. One example is risk in the domestic currency value of a foreign stock, where both the foreign exchange rate and the foreign stock price interact with each other. Another example is the hedging problem of a farmer who is uncertain about the output quantity and the selling price at harvest. Such problems can be classified as hedging problems with quantity uncertainty, which can be either traded (e.g.,

exchange rate) or nontraded (e.g., farmer's output or LSE's demand).⁴ This work focuses on the latter case, i.e., the hedging problems with nontraded quantity risk.

The hedging problem with nontraded quantity risk was first dealt with theoretically by McKinnon (1967). He recognized that risk aversion for a farmer consists of protecting himself from the output uncertainty as well as the market price uncertainty, and obtained the optimal position of short futures that minimizes the variance of profit. Assuming profit is given by $QS + H(F - S)$, where S is a spot price, Q is an uncertain output, H is the futures position, and F is a futures price ($F = E[S]$), the optimal minimum-variance hedge position is given by $H^* = \text{cov}(SQ, S)/\text{var}(S)$. Under the assumption of bivariate normality on spot price and quantity, McKinnon (1967) developed an explicit formula for H^* in terms of correlation coefficients and variances of price and quantity. The formula for the optimal hedging quantity showed that the correlation between (production) quantity uncertainty and price uncertainty is a fundamental feature of the problem. Other studies by Danthine (1978), Holthausen (1979), and Feder et al. (1980) extended the McKinnon's (1967) work to show that the nontraded quantity risk in the farmer's problem can be partially managed through the optimal choice of the input amount (e.g., the amount of input affects the output quantity) and futures position in a single optimization problem.

The limitation of the models based on McKinnon (1967) is that they only consider futures contracts as their hedging instruments. When a firm faces a multiplicative risk of price and quantity, its profit is nonlinear in price. In other words, the risk cannot be fully hedged by a forward or futures contract, which has a linear payoff structure. Moreover, as pointed out by Wong (2003), even without quantity risks, forward or futures contracts may not be enough because the nonlinearity may stem from using nonlinear marginal utility functions. Specifically, if the firm's preference

satisfies the reasonable behavioral assumption of prudence (Kimball 1990, 1993), the prudent firm will have a convex marginal utility function. Such a firm is more sensitive to low realizations of profit than high ones. To avoid the low realizations, the firm finds the asymmetric payoff profiles of options particularly useful. This is the case even with two independent sources of risk.

This idea was employed by Moschini and Lapan (1995), who included options in the optimal decisions for firms facing price, quantity, and basis risks under the constant absolute risk aversion utility. They solved for the optimal amount of straddle⁵ as well as futures in their hedging portfolio. They demonstrated that the nontraded quantity uncertainty can provide a rationale for the use of options.

Brown and Toft (2002) also considered a model accounting for multiplicative interaction of traded price risk and nontraded quantity risk, and explored the best exotic option for the customized hedging needs. Brown and Toft (2002) derived the optimal payoff function for the firm's value based on the payoff (as a function of price) from an exotic derivative. They demonstrated that this optimal exotic derivative better hedges the firm's price and quantity risks than the simple hedge, which uses a single "plain vanilla" option.

2.2. Hedging for Load-Serving Entities

Restructuring of electricity markets in recent years introduced inherent high volatility in electricity spot prices and resulted in the need for efficient hedging strategies for both generators and LSEs. However, the literature on this subject is scarce.

Vehviläinen and Keppo (2003) developed an integrated framework for the optimal management of price risk using a portfolio of electricity derivatives. Specifically, they provided a framework for the Monte Carlo simulation procedure for the optimal portfolio that maximizes the expected utility of terminal wealth. The accuracy of their model relied on the models for the price processes of derivative contracts.

The LSE's hedging problem of price and quantity risk under an expected utility maximization criterion

⁴ There is another stream of research on the optimal portfolio choice considering additive, but correlated, nontraded risk (Duffie and Zariphopoulou 1993, He and Pages 1993). Instead, the risk we consider in this work is a multiplicative function of traded and nontraded risks.

⁵ A straddle is a combination of a call and put at the same strike price. A straddle is used as a hedging instrument especially for protection against high volatility.

was addressed in the paper by Oum et al. (2006) mentioned above. Variants of this hedging problem using a VaR criterion have been addressed by Woo et al. (2004), Wagner et al. (2003), Kleindorfer and Li (2005), and Oum and Oren (2009). The VaR, which is defined as a maximum possible loss at a given confidence level, is a widely used risk measure in practice and has become a standard tool in risk management. However, the optimization problems with the VaR risk measure are hard to solve analytically without very restrictive assumptions, especially when price and quantity risks are considered. A more detailed discussion of VaR-based hedging for joint price and quantity exposure is given in Oum and Oren (2009). The effect of introducing weather derivatives into the volumetric hedging portfolio was recently explored by Lee and Oren (2009), but the main focus of that work is on the pricing of weather derivatives in a multisector economy where different sectors that are exposed to weather risk use a mix of weather derivatives and commodity-based instruments for hedging and risk sharing.

Näsäkkälä and Keppo (2005) also studied hedging of electricity cash flows with forward contracting strategies. Their formulation is basically a multiperiod extension of the McKinnon (1967) problem that determines the optimal hedge ratio in the presence of price and quantity risks. The quantity risk was modeled as a load estimate process, which represented the process of the estimates of load quantity at maturity. Static hedging strategies were considered because of concerns about transaction costs and illiquidity. When static hedging strategies are used, the agent faces at any point in time the decision of whether to hedge based on the current load estimate or wait for new information. They found the optimal hedging ratio and timing that minimizes the variance of the portfolio's cash flow.

It is also worth mentioning a few papers on portfolio optimization for the supply side in the electricity market. Producers, especially ones who own hydroelectric plants or those that sell load-following fixed-price contracts, also face severe volumetric risk because their production capacities or supply obligation highly depends on weather condition such as precipitation and temperature. Their operational decision regarding when and how much to produce

should be combined with a hedging strategy in the spot, forward, and options market, but the difficulties arise because an operational decision for one period affects decisions for the later periods. Fleten et al. (1999), Gussow (2001), Herzog (2002), and Unger (2002) deal with such problems and solve them using multiperiod stochastic dynamic programming.

In positioning this paper it is worth noting two major dimensions that differentiate much of the work on hedging in the context of electricity market. One dimension concerns the optimization criteria used, that is, expected utility versus VaR. The second dimension concerns the way of constructing an implementable hedging portfolio. One approach employed, for instance, by Kleindorfer and Li (2005) is to start with a basket of instruments that are offered in the market and derive the optimal mix given the specific optimization criteria and budget constraints. This approach lends itself to direct practical implementation. The alternative approach employed in this paper and its predecessors (Oum et al. 2006, Oum and Oren 2009) starts with an arbitrary continuous payoff function that is optimized under proper criteria and self-financing constraints. The solution of such problems provides valuable insight regarding the structure of an ideal hedging portfolio. An implementable approximation of the optimal portfolio can then be derived using approximate replication schemes. Such replication, however, may yield a portfolio that is suboptimal given the discrete choices of available instruments.

3. Optimal Static Hedging in a Single-Period Setting

In this section, we reproduce, for completeness, results from Oum et al. (2006) upon which we build the optimal timing extension described in the subsequent section. The problem is solved in two steps. First we solve for the payoff of the optimal hedging portfolio, and then we replicate that payoff function with a portfolio of standard instruments.

3.1. Finding the Optimal Hedge Payoff Function

Consider a cost-free hedging portfolio consisting of electricity derivatives, constructed at time 0, whose payoff at time 1, $x(p)$, is a function of the spot price p at time 1. The hedging portfolio may also

include money market accounts, allowing the LSE to finance hedging instruments through loans payable at time 1. Let $y(p, q)$ be the LSE’s profit from serving the customers’ demand q at the fixed retail rate r at time 1. Then, the hedged profit $Y(p, q, x(p))$ —total profit including the net payoffs of the hedging portfolio—is given by

$$Y(p, q, x(p)) = y(p, q) + x(p) = (r - p)q + x(p). \quad (1)$$

The LSE’s risk preference is characterized by a concave utility function $U(\cdot)$ defined over the total profit $Y(\cdot)$ at time 1. The LSE’s beliefs on the realization of spot price p and load q are characterized by a joint probability function $f(p, q)$ for positive p and q , which is defined on the probability measure P . On the other hand, let Q be a risk-neutral probability measure based on which the hedging instruments are priced, and let $g(p)$ be the probability density function of p under Q . Because the electricity market is incomplete, there may exist infinitely many risk-neutral probability measures. In this paper, it is assumed that a specific measure, Q , was picked according to some suitable criteria.⁶

Then, the formulation of the optimal static hedging problem is as follows:

$$\begin{aligned} \max_{x(p)} & E[U[Y(p, q, x(p))]] \\ \text{s.t.} & E^Q[x(p)] = 0, \end{aligned} \quad (2)$$

where $E[\cdot]$ and $E^Q[\cdot]$ denote expectations under the probability measures P and Q , respectively. The constraint requires the “manufacturing” cost of the portfolio (ignoring transaction costs) to be zero under a constant risk-free interest rate. This zero-cost constraint implies that purchasing derivative contracts, which are priced at their expected payoff with respect to the risk-neutral probability measure, may be financed from selling other derivative contracts or through money market accounts. In other words,

⁶In a complete market, there is a unique risk-neutral probability measure such that the no-arbitrage prices of hedging instruments equals the expected value of their payoff with respect to that probability measure. When the market is incomplete, as in our case, the risk-neutral probability measure is not unique, and there are many proposed criteria for choosing a suitable risk-neutral measure. See Xu (2006) for this subject.

under the assumption that there is no limit on the possible amount of instruments to be purchased and money to be borrowed, the model finds a portfolio from which the LSE obtains the maximum expected utility over total profit.

The Lagrangian function for the above constrained optimization problem is given by

$$\begin{aligned} L(x(p)) &= E[U(Y(p, q, x(p)))] - \lambda E^Q[x(p)] \\ &= \int_{-\infty}^{\infty} E[U(Y) | p] f_p(p) dp - \lambda \int_{-\infty}^{\infty} x(p) g(p) dp, \end{aligned}$$

with a Lagrange multiplier λ and the marginal density function $f_p(p)$ of p under P .

Differentiating $L(x(p))$ with respect to $x(\cdot)$ results in

$$\frac{\partial L}{\partial x(p)} = E\left[\frac{\partial Y}{\partial x} U'(Y) \mid p\right] f_p(p) - \lambda g(p) \quad (3)$$

by the Euler equation. Setting (3) to zero and substituting $\partial Y / \partial x = 1$ from (1) yields the first-order condition for the optimal solution $x^*(p)$ as follows:

$$E[U'(Y(p, q, x^*(p))) | p] = \lambda^* \frac{g(p)}{f_p(p)}. \quad (4)$$

Here, the value of λ^* should be the one that satisfies the zero-cost constraint (2).

In the remainder of this paper we will restrict ourselves to a mean-variance utility function that is a commonly used approximation to more general expected utility functions. For a discussion regarding when the use of a mean-variance expected utility function is justified, see the recent paper by Morone (2008).

PROPOSITION 1. *For an agent who maximizes mean-variance expected utility of profit,*

$$E[U(Y)] = E[Y] - \frac{1}{2} a \text{Var}(Y),$$

the optimal solution $x^(p)$ to problem (2) is given as*

$$\begin{aligned} x^*(p) &= \frac{1}{a} \left(1 - \frac{g(p)/f_p(p)}{E^Q[g(p)/f_p(p)]} \right) - E[y(p, q) | p] \\ &\quad + E^Q[E[y(p, q) | p]] \frac{g(p)/f_p(p)}{E^Q[g(p)/f_p(p)]}. \end{aligned} \quad (5)$$

Moreover, suppose the joint distributions of p and q are bivariate lognormal distributions as follows:

$$\text{Under } P, \log p \sim N(m_1, s^2), \quad \log q \sim N(m_q, u_q^2),$$

$$\text{Corr}(\log p, \log q) = \phi;$$

$$\text{Under } Q, \log p \sim N(m_2, s^2).$$

Then,

$$x^*(p) = \frac{1}{a}(1 - B_1(p)) - B'_2(p) + B'_3 B_1(p), \quad (6)$$

where

$$B_1(p) = p^{((m_2 - m_1)/s^2)} \cdot \exp\left(-\frac{(m_1 - m_2)(m_1 - 3m_2)}{2s^2}\right),$$

$$B'_2(p) = (r - p) \cdot \exp\left(m_q + \phi \frac{u_q}{s}(\log p - m_1) + \frac{1}{2}u_q^2(1 - \phi^2)\right),$$

$$B'_3(p) = r \cdot \exp\left(m_q + \phi \frac{u_q}{s}(m_2 - m_1) + \frac{1}{2}u_q^2(1 - \phi^2) + \frac{1}{2}\phi^2 u_q^2\right) - \exp\left(m_2 + m_q + \phi \frac{u_q}{s}(m_2 - m_1) + \frac{1}{2}u_q^2(1 - \phi^2) + \frac{1}{2}\left(\phi \frac{u_q}{s} + 1\right)^2 s^2\right).$$

The proof is contained in Oum et al. (2006).

COROLLARY 1. Under the assumption of $P \equiv Q$, the optimal payoff function under the mean-variance expected utility becomes

$$x^*(p) = E[y(p, q)] - E[y(p, q) | p]. \quad (7)$$

PROOF. If $P \equiv Q$, then $g(p)/f_p(p) = 1$. Then, Equation (5) reduces to Equation (7). \square

The assumption that $P \equiv Q$ was empirically justified by Audet et al. (2004) and Koekebakker and Ollmar (2005) for the Nordic electricity forward market.⁷ The first term, $E[y(p, q)]$, in Equation (7) is a constant, and the second term, $E[y(p, q) | p]$, is the expected profit given the value of the spot price. The formula implies that the optimal payoff is one that

⁷ Although we adopt the assumption that $P = Q$, for analytical convenience, we note that the existence of a risk premium in electricity is a debatable topic, and a recent paper by Lucia and Torró (2008) based on a decade of data on short-term future prices in Nordpool reaches the conclusion that there exists a positive risk premium in that market that varies seasonally. According to these findings, the risk premium is zero during the summer, but it is significant in the winter, and particularly high when reservoir levels are unexpectedly low. Sensitivity analysis of the optimal hedging strategy with regard to the risk premium, under the assumption that P and Q only differ in the mean but have the same variance, is provided in Oum et al. (2006) and will be omitted in this paper.

levelizes the conditional expectation of hedged profit across spot prices p . This is because maximizing the mean-variance objective function given the zero-cost constraint and $P \equiv Q$ is the same as just minimizing a variance of hedged profits.⁸ In fact, (7) means that the optimal portfolio removes all the uncertainty in the profit that is correlated with price.

3.2. Replication of Exotic Payoffs

As noted by Oum et al. (2006, 2009), the continuous optimal payoff function $x(p)$ can be replicated by a portfolio consisting of a spectrum of put and call options with continuous strike prices, forwards, and bonds. Such replication is based on the work of Carr and Madan (2001), who showed that any twice continuously differentiable function $x(p)$ can be written in the following form:

$$x(p) = [x(s) - x'(s)s] + x'(s)p + \int_0^s x''(K)(K - p)^+ dK + \int_s^\infty x''(K)(p - K)^+ dK,$$

for an arbitrary positive s . The replication is obtained by setting s to the forward price F at time 0, resulting in

$$x(p) = x(F) \cdot 1 + x'(F)(p - F) + \int_0^F x''(K)(K - p)^+ dK + \int_F^\infty x''(K)(p - K)^+ dk. \quad (8)$$

The terms 1 , $(p - F)$, $(K - p)^+$, and $(p - K)^+$ on the right-hand side of Equation (8) can be interpreted as the unit payoffs of a bond, forward, European put option with strike price K , and a European call option with strike price K , respectively. The corresponding multipliers $x(F)$, $x'(F)$, $x''(K)dK$, and $x''(K)dK$ are the respective quantities of these instruments in a portfolio that achieves exact replication of $x(p)$. In other words, exact replication can be obtained from a long cash position of size $x(F)$, a long forward position of size $x'(F)$, long positions of size $x''(K)dK$ in puts struck at K , for a continuum of $K < F$, and a long position of size $x''(K)dK$ in calls struck at K , for a continuum of $K > F$. Note that unless the optimal payoff function is linear, the

⁸ This kind of hedging is also considered in Näsäkkälä and Keppo (2005): mean-variance hedging reduces to variance minimization when the pricing measure equals to the physical measure because they consider only forward contracts, which have zero expected value before delivery.

optimal strategy involves purchasing (or selling short) a spectrum of both call and put options with a continuum of strike prices. This result proves that to hedge price and quantity risks together, LSEs should purchase a portfolio of options. The strike prices of call options effectively work as “buying” price caps on each increment of load.

In practice, electricity derivatives markets, as any derivatives markets, are incomplete because the market does not offer options for the full continuum of strike prices, but typically only a limited number of discrete strike prices are traded. Thus, practically we can only approximate the payoff function $x(p)$ using available discrete strike prices. A discussion of such approximation methods and the error they induce is contained in Oum et al. (2006, 2009) and will not be repeated here. However, in presenting the results for our numerical examples of this paper, we will illustrate the replicating portfolio under discrete strike prices.

3.3. Example I

As a reference case for the subsequent analysis and numerical result, we first present an illustrative example of the static single-period hedging strategy discussed above.

Consider a hypothetical LSE that is characterized by the following assumptions:

- Price is distributed lognormally with parameters $m_1 = 4$ and $s = 0.7$ in both the real-world and risk-neutral world: $\log p \sim N(4, 0.7^2)$ in P and Q . The expected value and the standard deviation of price p under this distribution is \$70 per megawatt hour and \$56 per megawatt hour, respectively.
- The LSE charges a flat retail rate $r = \$120$ per megawatt hour to its customers.
- Load is lognormally distributed with parameter $m = 7.99$ and $u = 0.2$.

Figure 1(a) illustrates the optimal payoff functions obtained under the mean-variance expected utility by varying correlations. When correlation is zero, the payoff function becomes linear, meaning that a single forward contract with a linear payoff is sufficient. However, when there is positive correlation, the optimal payoff demonstrates nonlinearity, telling us that there is a need for derivatives other than forward contracts.

Figure 1(b) shows the distributions of profits with price hedging⁹ and with price and quantity hedging.¹⁰ First, observe that both hedges reduce the variance of profit relative to the unhedged profit. Second, observe that price and quantity hedge chops off the left tail of the profit distribution after price hedging. This implies that the LSE can protect itself against rare but detrimental events by hedging quantity risk. Moreover, the LSE can benefit from the longer right tail of the profit distribution after quantity hedging.

The replication strategy of the optimal payoff function in Figure 1(a) is shown in Figure 1(c). Assuming that options are available for strike prices of F (the forward price) and each increment and decrement of \$10 starting from F , Figure 1(c) shows the number of each of the contracts that should be purchased. The figure also shows that the forward contract covers slightly less than the expected demand (3,000 MWh), whereas call options are used to cover the incremental demand corresponding to high spot prices. Figure 1(d) confirms that the total payoff from our replication is very close to the optimal payoff function $x(p)$ that we want to replicate.

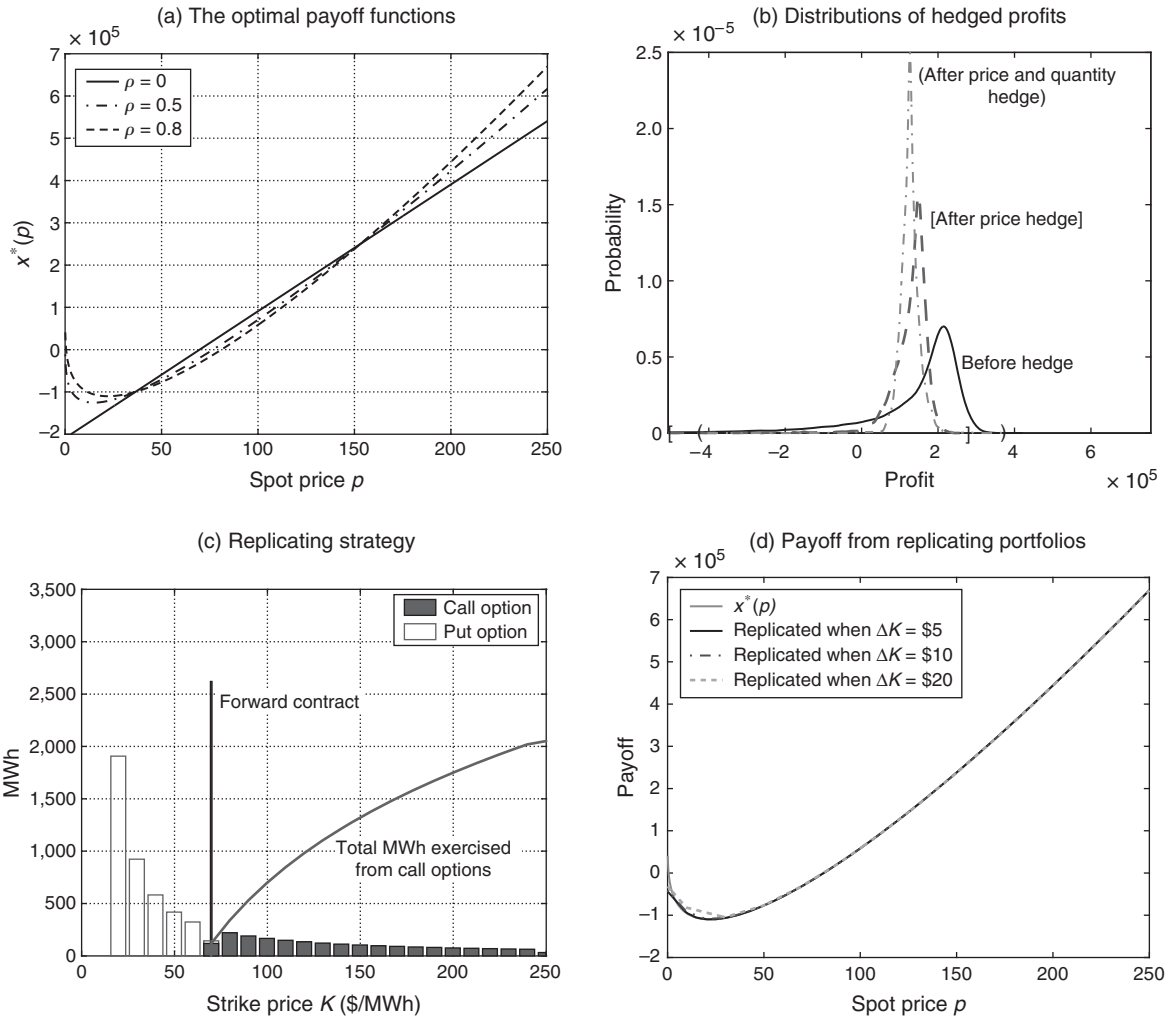
4. Timing of a Static Hedge in a Continuous-Time Setting

Let T be the delivery period and maturing date of the hedging instruments. We will assume that all the hedging instruments for the delivery period T are contracted at the same time τ . Contracting earlier reduces the risk by locking in the price of the contracts, whereas delaying the contracting enables more profitable hedging by exploiting more information that becomes available as we approach maturity. We will assume that the optimal hedging time is determined at time t_0 based on the information available at that time, but the composition of the hedging portfolio is determined at the hedging time τ based on the forward price and the information about spot price and load at the time. It should be noted that this is not a dynamically consistent strategy because

⁹ “Price hedge” here means that we add the optimal payoff function obtained under the assumption of no quantity risk. This is in fact equivalent to buying forward contracts for the average load quantity.

¹⁰ “Price and quantity hedge” refers to the optimal payoff function that we obtained in this paper.

Figure 1 Hedging with Mean-Variance Utility (Minimizing Variance) Under Bivariate Lognormal Distribution for Price and Load



Note. $\log p \sim N(4, 0.7^2)$, $\log q \sim N(7.99, 0.2^2)$; $\text{Corr}(\log p, \log q) = \rho = 0.8$, $(P = Q)$.

the optimal hedging time determined at t_0 might no longer be optimal at time $t_1 > t_0$ because more information becomes available at time t_1 . Nevertheless, we will assume that the contracting time is chosen irreversibly at time t_0 . Such open-loop optimization is a quite common second-best approach due to the difficulty of developing a closed-loop feedback strategy, and it can be viewed as a first step in exploring the sensitivity of optimal static hedging outcomes with respect to contract timing.

4.1. Mathematical Formulation

Let $\{p_t\}_{t \in [0, T]}$ be a process of forward price for delivery at time T and $\{q_t\}_{t \in [0, T]}$ be a process for load estimate

for period T calculated at time t . Assume that the forward price and load estimate processes evolve according to the following Ito processes:

$$dp_t = p_t(\mu_p(t)dt + \sigma_p(t)dB_t^1), \quad (9)$$

$$dq_t = q_t(\mu_q(t)dt + \sigma_p(t)dB_{qt}^1 + \sigma_q(t)dB_t^2), \quad (10)$$

where B_t^1 and B_t^2 are independent Wiener processes. Then, p_T and q_T denote the spot price and demand at time T . Although the time-dependent volatilities in the above processes provide wide latitude for capturing behavior of real market data, it is out of the scope of this paper to verify that this model fits the real price and quantity movements or to estimate the model parameters from such data. On the other hand,

we note that this model generalizes the specification estimated by Audet et al. (2004) based on NordPool market data and used in the paper by Näsäkkälä and Keppo (2005) in calculating the optimal timing of a simple forward contract.

The optimal hedging timing determined at time t_0 for a optimal hedge is characterized by the following maximization problem:

$$\max_{\tau \geq t_0} E_{t_0} [U((r - p_T)q_T + x_\tau(p_T))], \quad (11)$$

$$\begin{aligned} \text{where } x_\tau(p_T) &= \arg \max_{x(p_T)} E_\tau [U((r - p_T)q_T + x(p_T))] \\ \text{s.t. } E_\tau^Q [x(p_T)] &= 0. \end{aligned} \quad (12)$$

In this formulation, $x_\tau(p_T)$ denotes the payoff from the optimal portfolio to be constructed at time τ . Thus, the formulation finds a time τ^* ; hedging at that time maximizes the expected utility of the optimally hedged profit.

Throughout this section, it is assumed that the physical probability measure and risk-neutral probability measure are the same, and we restrict ourselves to the mean-variance utility function as mentioned earlier. Consequently, because of the zero-cost constraint, maximizing a mean-variance objective function is reduced to minimizing the variance of the hedged profit. The formula for $x_\tau(p_T)$ can be obtained from the results of the §3 (see Equation (7)). Thus, the problem becomes a single-variable unconstrained optimization problem that can be easily solved numerically.

4.2. Finding the Optimal Payoff Function at Contracting Time

PROPOSITION 2. *Suppose $\{p_t\}_{t \in [0, T]}$ and $\{p_t\}_{t \in [0, T]}$ follow Ito processes given (9) and (10). Assuming $P = Q$, then $x_\tau(p_T)$ that solves (12) for a mean-variance utility function is obtained as follows:*

$$x_\tau^*(p_T) = B_\tau(p_T - r)p_T^{A_\tau} p_\tau^{-A_\tau} q_\tau + rC_\tau q_\tau - D_\tau p_\tau q_\tau, \quad (13)$$

where

$$\begin{aligned} A_\tau &= \int_\tau^T b_t d_t dt / \int_\tau^T b_t^2 dt, \\ B_\tau &= \exp \left(\int_\tau^T c_t dt - A_\tau \int_\tau^T a_t dt + \frac{1}{2} \int_\tau^T (d_t^2 + e_t^2) dt \right. \\ &\quad \left. - \frac{1}{2} \left(\int_\tau^T b_t d_t dt \right)^2 / \int_\tau^T b_t^2 dt \right), \end{aligned} \quad (14)$$

$$C_\tau = \exp \left(\int_\tau^T (c_t + \frac{1}{2} d_t^2 + \frac{1}{2} e_t^2) dt \right),$$

$$D_\tau = \exp \left(\int_\tau^T (a_t + c_t + \frac{1}{2} b_t^2 + \frac{1}{2} d_t^2 + \frac{1}{2} e_t^2 + b_t d_t) dt \right).$$

PROOF. The proof is given in the appendix.

Equation (13) is the payoff of the optimal portfolio to be constructed when hedging at time τ . The optimal portfolio incorporates the information of the forward price and load estimate available at the hedging time τ .

4.3. Determining the Optimal Hedging Time

With the assumption $P = Q$ and the zero-cost constraint $E_\tau[x_\tau] = 0$, maximizing (11) reduces to minimizing

$$\Pi(\tau) \equiv \text{Var}((r - p_T)q_T + x_\tau^*(p_T)).$$

Given x_τ^* obtained in §4.2, the problem (11) is in fact an unconstrained optimization problem with a single decision variable in the interval $[0, T]$. Once Π is obtained as a function of τ , the problem is solvable numerically even though $\Pi(\tau)$ is neither convex or concave. This section is concluded with the calculation of $\Pi(\tau)$:

$$\begin{aligned} \Pi(\tau) &= \text{Var}(x_\tau^*(p_T)) + 2\text{cov}((r - p_T)q_T, x_\tau^*(p_T)) \\ &\quad + \text{Var}((r - p_T)q_T), \end{aligned}$$

where

$$x_\tau^*(p_T) = B_\tau p_T^{A_\tau+1} p_\tau^{-A_\tau} q_\tau - rB_\tau p_T^{A_\tau} p_\tau^{-A_\tau} q_\tau + rC_\tau q_\tau - D_\tau p_\tau q_\tau.$$

Each term of $\Pi(\tau)$ is calculated as a function of τ as follows (for notational convenience, the subscript τ for A_τ, B_τ, C_τ , and D_τ is omitted):

$$\begin{aligned} \text{Var}(x_\tau^*(p_T)) &= E[x_\tau^*(p_T)^2] \\ &= B^2 E[p_T^{2A+2} p_\tau^{-2A} q_\tau^2] + r^2 B^2 E[p_T^{2A} p_\tau^{-2A} q_\tau^2] \\ &\quad + r^2 C^2 E[q_\tau^2] + D^2 E[p_\tau^2 q_\tau^2] \\ &\quad - 2rB^2 E[p_T^{2A+1} p_\tau^{-2A} q_\tau^2] + 2rBCE \\ &\quad \cdot [p_T^{A+1} p_\tau^{-A} q_\tau^2] - 2BDE [p_T^{A+1} p_\tau^{-A+1} q_\tau^2] \\ &\quad - 2r^2 BCE [p_T^A p_\tau^{-A} q_\tau^2] + 2rBDE [p_T^A p_\tau^{-A+1} q_\tau^2] \\ &\quad - 2rCDE [p_\tau q_\tau^2], \end{aligned}$$

$$\begin{aligned}
 & \text{cov}((r-p_T)q_T, x_T^*(p_T)) \\
 &= E[(r-p_T)q_T x_T^*(p_T)] \\
 &= rBE[p_T^{A+1}q_T p_T^{-A}q_T] - r^2BE[p_T^A q_T p_T^{-A}q_T] \\
 &+ r^2CE[q_T q_T] - rDE[q_T p_T q_T] - BE[p_T^{A+2}q_T p_T^{-A}q_T] \\
 &+ rBE[p_T^{A+1}q_T p_T^{-A}q_T] - rCE[p_T q_T q_T] + DE[p_T q_T p_T q_T], \\
 & \text{Var}((r-p_T)q_T) \\
 &= E[(rq_T - p_T q_T)^2] - E[rq_T - p_T q_T]^2 \\
 &= r^2E[q_T^2] - 2rE[p_T q_T^2] + E[p_T^2 q_T^2] \\
 & \quad - (rE[q_T] - E[p_T q_T])^2.
 \end{aligned}$$

The expectation terms were calculated using

$$\begin{aligned}
 & E[p_T^\alpha q_T^\beta p_T^\gamma q_T^\delta] \\
 &= p_0^{\alpha+\gamma} \cdot q_0^{\beta+\delta} \cdot \exp\left(\int_0^T (\alpha a_t + \beta c_t) dt\right) \\
 & \quad \cdot \exp\left(\int_0^T \left(\frac{1}{2}(\alpha b_t + \beta d_t)^2 + \frac{1}{2}\beta^2 e_t^2\right) dt\right) \\
 & \quad \cdot \exp\left(\int_0^T \left(\gamma a_t + \delta c_t \right. \right. \\
 & \quad \quad \left. \left. + \frac{1}{2}((\alpha + \gamma)b_t + (\beta + \delta)d_t)^2 \right. \right. \\
 & \quad \quad \left. \left. + \frac{1}{2}(\beta + \delta)^2 e_t^2\right) dt\right)
 \end{aligned}$$

from

$$\begin{aligned}
 p_T^\alpha &= p_0^\alpha \exp\left(\int_0^T \alpha a_t dt + \int_0^T \alpha b_t dB_t^1\right), \\
 qq_T^\beta &= p_0^\beta \exp\left(\int_0^T \beta c_t dt + \int_0^T \beta d_t dB_t^1 + \int_0^T \beta e_t dB_t^2\right), \\
 p_T^\gamma &= p_0^\gamma \exp\left(\int_0^T \gamma a_t dt + \int_0^T \gamma b_t dB_t^1\right), \\
 q_T^\delta &= p_0^\delta \exp\left(\int_0^T \delta c_t dt + \int_0^T \delta d_t dB_t^1 + \int_0^T \delta e_t dB_t^2\right).
 \end{aligned}$$

4.4. Example II

We now illustrate the optimal hedging timing problem with a concrete example. The example assumes that the maturity of the portfolio is one year from now. Base values of the parameters are set according to the empirical estimates of Audet et al. (2004), which were also used by Näsäkkälä and Keppo (2005). Specifically, we set $\mu_p(t) = 0$, and $\sigma_p(t) = e^{\psi(T-t)}\sigma$, where σ is the spot volatility and ψ is a mean-reversion rate of the spot price process, i.e., a rate at

which forward volatility is discounted from the spot volatility. We also set $\mu_q(t) = 0$. In addition, $\sigma_{pq}(t)$ and $\sigma_q(t)$ are assumed to be constant, so as to have constant load volatility and correlation:

$$\sigma_L = \sqrt{\sigma_{pq}^2 + \sigma_q^2}, \quad \phi = \sigma_{pq} \sigma_L.$$

The resulting process is then

$$\begin{aligned}
 \frac{dp_t}{p_t} &= e^{-\psi(T-t)}\sigma dB_t^1, \\
 \frac{dq_t}{q_t} &= \phi\sigma_L dB_t^1 + \sqrt{1-\phi^2}\sigma_L dB_t^2.
 \end{aligned} \tag{15}$$

The forward price and load estimate for a month one year later are assumed to be 20 euro/MWh and 1,000 MWh, respectively. The following table summarizes the base values of the parameters.

Parameter	T	r	p_0	q_0	ψ	σ	σ_L	ϕ
Value	1	40	20	1,000	4.02	0.7	0.1	0.7

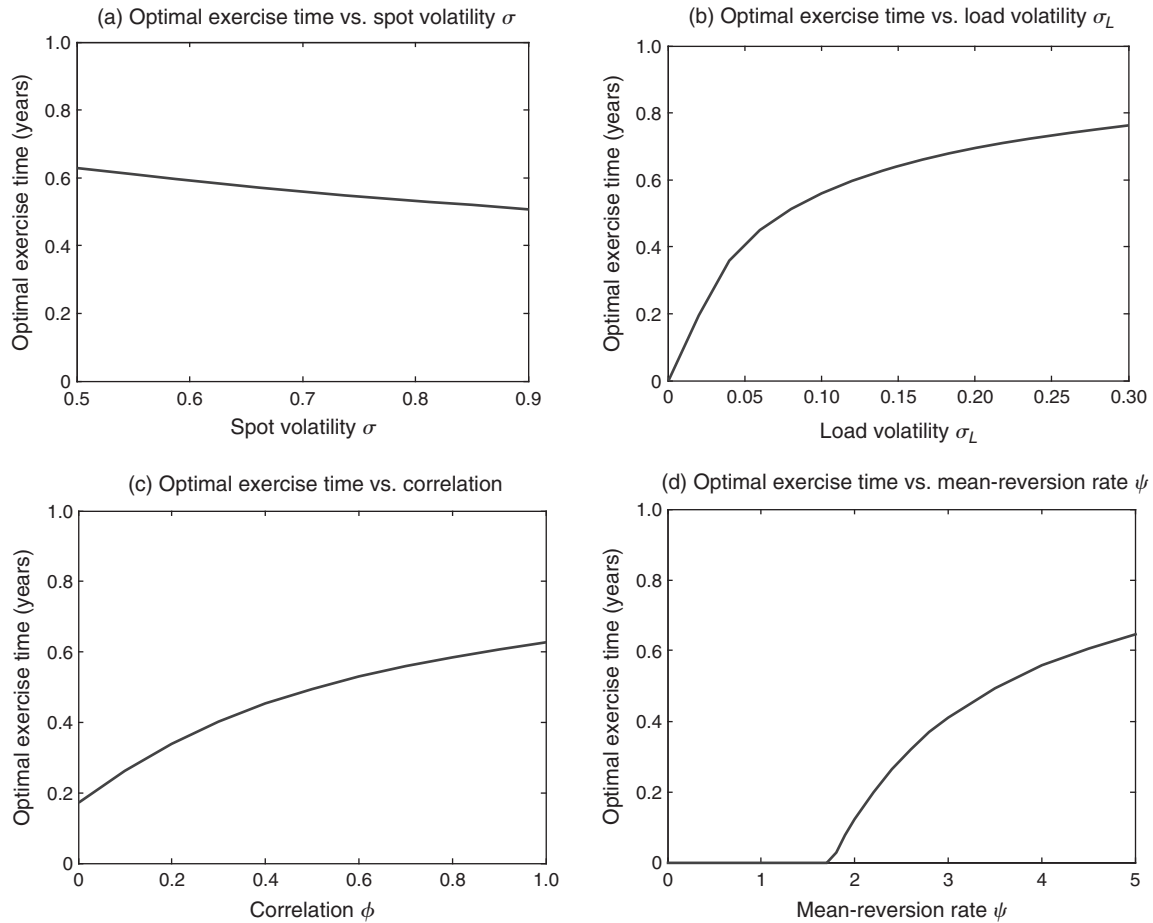
To study how the optimal hedging time is affected by various parameters, a sensitivity analysis of optimal hedging time with respect to parameter values is illustrated in Figure 2.

Figure 2(a) plots the optimal hedging time against the spot price volatility σ , and it shows that a higher spot volatility favors earlier hedging. Intuitively, a higher spot volatility increases uncertainties in the future price information, which justifies locking in the price of hedging contracts earlier.

Figure 2(b) plots the optimal hedging time against the load volatility σ_L . It shows that a higher volatility in the load estimate postpones the optimal hedging time, confirming the intuition that the inaccuracy in the load estimate favors delaying the hedging time so as to obtain more information.

Figure 2(c) plots the optimal hedging time against the correlation between forward price and load estimate. It shows that a lower correlation makes earlier hedging more favorable. To explain this phenomenon, we first note from the above that price and quantity volatility push the optimal contract time in opposite directions. We also note that as correlation decreases, our ability to hedge quantity risk with price-based instruments diminishes while the adverse effect of quantity volatility on profit uncertainty decreases. In

Figure 2 Optimal Hedging Time vs. Other Parameter Values



the extreme case where price and quantities are independent, the curve $x^*(p)$ becomes linear, indicating that the optimal portfolio consists of a fixed number of at-the-money forward contracts. Consequently, lower correlation between price and quantity (with the same individual volatilities) makes the hedging portfolio look more like a pure price hedge, which favors an earlier contracting time.

Figure 2(d) plots the optimal hedging time against the mean-reversion rate of the spot price. The figure shows that an increase in the mean-reversion rate of the spot price postpones the hedging time, because a higher mean-reversion rate of the spot price decreases the volatility of forward prices, so it will not be as risky to postpone the hedging time.

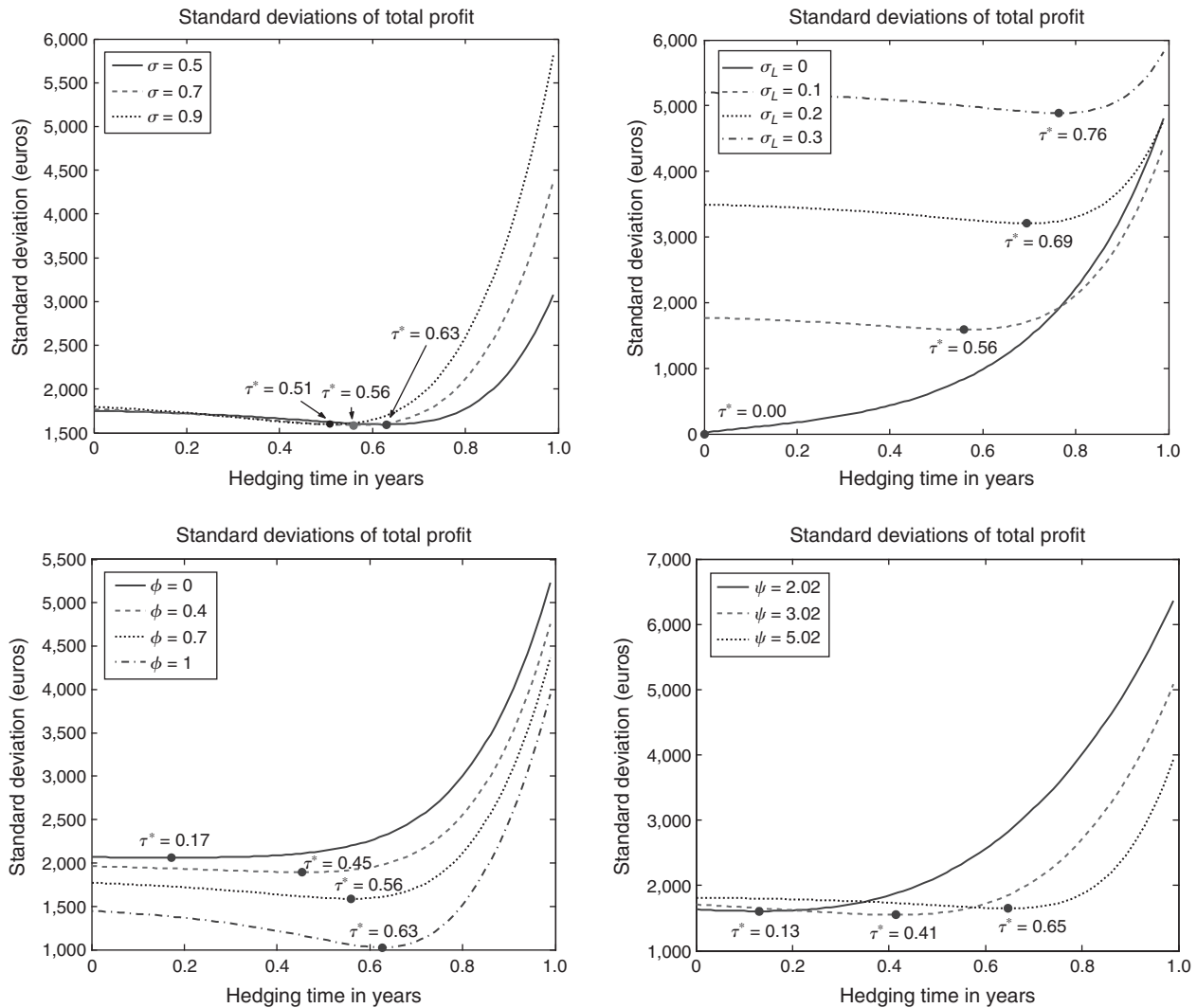
Figure 3 illustrates the variance of the optimally hedged profit as a function of the hedging time. We

note that hedging at time 0 versus the optimal time τ^* makes little difference in the variance of hedged profit in most cases. However, the variance of profit increases rapidly if hedging is delayed beyond the optimal time.

Figure 3 also shows how the level of uncertainties changes with respect to the changes in σ , σ_L , ϕ , and ψ . The data displayed in the figure indicate that the profit uncertainty increases with the increases in spot and load volatility, and it decreases with the mean-reversion rate and correlation coefficient.

It is also noteworthy that hedging at the optimal time may not make any difference in the variance of hedged profit even for different parameters such as volatility and mean-reversion rate of spot price. In other words, the increased uncertainty from higher

Figure 3 Standard Deviation of Hedged Profit vs. Hedging Times



volatility in the forward price can be overcome by the optimal choice of hedging time.

Figure 4 compares the distributions of profits at delivery time when the hedging portfolio is purchased at time 0, at the optimal hedging time (0.56), and at time (0.9) close to delivery time. It also confirms that earlier hedging does not increase profit risk very much as compared to the optimal hedging time, but late hedging can have adverse consequences.

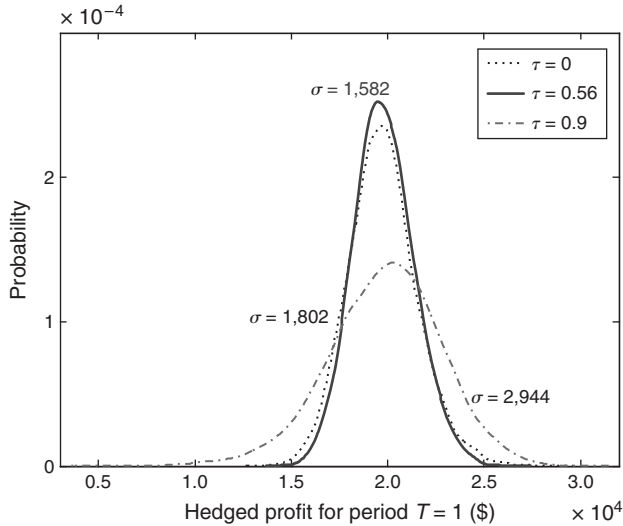
Finally, the optimal hedging strategy at time 0, under the base values of the parameters, is illustrated in Figure 5, which shows the optimal payoff function and its approximate replication when strike prices

change in \$5 increments, if the hedging portfolio was constructed at time 0 (in our analysis, we assume that the hedging portfolio can be optimized at contracting time).

5. Conclusion

This paper developed a method of mitigating volumetric risk that LSEs and marketers of default service contracts face in providing their customers' load-following service at fixed or regulated prices while purchasing electricity or facing an opportunity cost at volatile wholesale prices. Exploiting the

Figure 4 Distributions of Optimally Hedged Profits for Different Hedging Times



Note. The optimal hedging time based on the information given at time 0 is $\tau = 0.56$.

inherent positive correlation and multiplicative interaction between wholesale electricity spot price and demand volume, we developed a hedging strategy for the LSE’s retail positions (which is in fact a short position on unknown volume of electricity) using electricity standard derivatives such as forwards, calls, and puts.

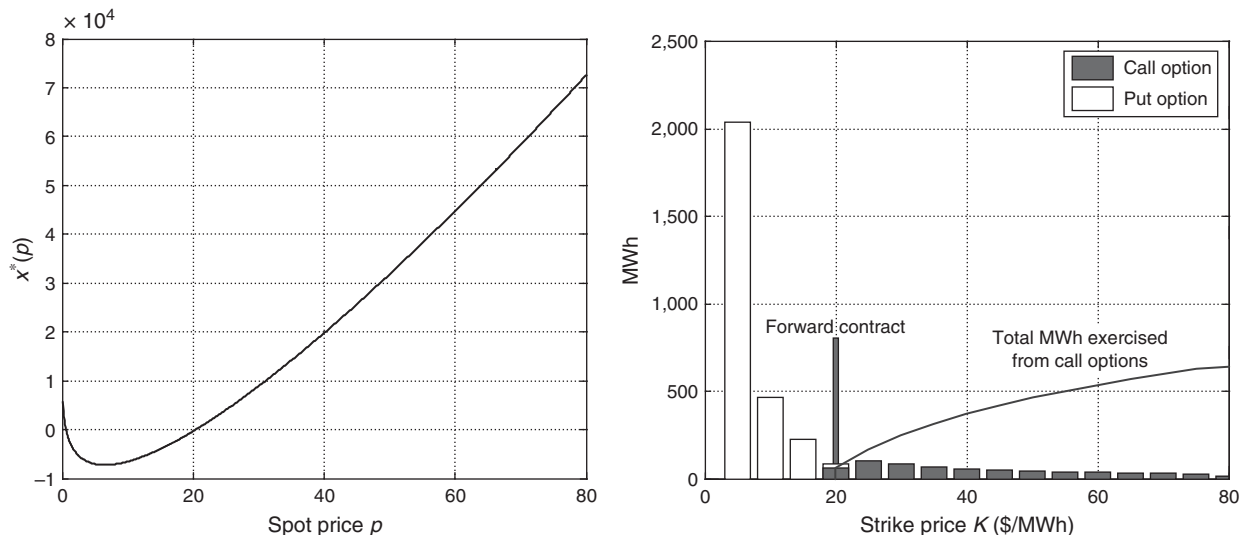
The optimal hedging strategy was determined based on expected utility maximization, which has

been used in the hedging literature to deal with non-tradable risk. We derived an optimal payoff function that represents the payoff of the optimal costless exotic option as a function of price. We then showed how the optimal exotic option can be replicated using a portfolio of forward contracts and European options.

The examples demonstrated how call and put options can improve the hedging performance when quantity risk is present, compared to hedging with forward contracts alone. Although at present the liquidity of electricity options is limited, the use of call options has been advocated in the electricity market design literature as a tool for resource adequacy, market power mitigation, and spot volatility reduction. The result of this paper contributes to a better understanding of how options can be utilized in hedging the LSE’s market risk and hopefully will increase their use and their liquidity in electricity markets.

This paper extended previous work by considering the optimal timing of a hedging portfolio as well as the co-optimization of the portfolio mix taking account of the timing. For mean-variance expected utility, we solved for the optimal hedging time under the classical assumption regarding the stochastic processes governing forward price and load estimate. Because the primary objective of this paper was to investigate the sensitivity of the hedging strategy and gains from hedging with respect to contract timing,

Figure 5 The Optimal Payoff Function and Its Replication When the Hedging Portfolio Is Constructed at Time 0



we employed a simplistic assumption that facilitates an analytic solution amenable to sensitivity analysis and intuitive interpretation. Specifically, we focused on a single-period setting and assumed an open-loop static strategy where the hedging portfolio is selected at a single specific time within the period, for exercise at the end of the period when the commodity is to be delivered. Furthermore, the optimal contracting time is irreversibly determined at the beginning of the period. Notably, such a strategy is not dynamically consistent because new information obtained after the contracting time that has been determined may indicate a better choice of contracting time. A more complete treatment of the problem would involve a stochastic dynamic programming formulation where the choice of optimal contracting can be formalized as an optimal stopping time problem.

The illustrative example presented shows that generally there is a critical time beyond which the uncertainty in profit increases sharply, whereas the uncertainty remains relatively constant before this critical time. Sensitivity analysis results indicate that the optimal hedging time gets closer to the delivery period if the positive correlation between the forward price and load estimate is higher, and if the load-estimate volatility is higher. It is also observed that delaying the hedging time past the optimum time can be very risky, whereas premature hedging makes little difference as compared with hedging at the optimal time. This suggests that in practice one should err by hedging early rather than taking the chance of being too late.

The model presented in this paper determined the best hedging portfolio assuming that the LSE has unlimited borrowing capability. In practice, credit limits can become an impeding factor in purchasing the optimal hedging portfolio. An LSE may not be able to borrow enough upfront money to finance the options contracts. Therefore, a credit limit constraint, which limits the amount of money that can be borrowed to construct the portfolio, needs to be considered in future extensions of our model. A dynamic hedging strategy rather than the static approach adopted in this paper is likely to improve the hedging performance and should be considered in future extension of this work as well.

Finally, we note that at this point in time the practical value of our results is limited because of the

unavailability of electricity derivatives with a full spectrum of strike prices. As more and more marketers or traders are coming into the electricity market, making electricity instruments more liquid, the hedging strategies developed in this paper will become more relevant in practice.

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Appendix

PROOF OF PROPOSITION 2. Recall that, under $P = Q$, the optimal payoff function for the mean-variance optimizer was given by the following formula (see (7)):

$$x^*(p) = E[y(p, q)] - E[y(p, q) | p],$$

where $y(p, q) = (r - p_T)q_T$. The payoff $x_\tau(p_T)$ at (13) is then obtained by taking conditional expectations at time τ , instead of at time 0:

$$x_\tau^*(p_T) = E_\tau[(r - p_T)q_T] - (r - p_T)E_\tau[q_T | p_T]. \quad (16)$$

Given Equations (9) and (10), Ito's formula obtains p_T and q_T as follows:

$$p_T = p_\tau \exp \left\{ \int_\tau^T a_t dt + \int_\tau^T b_t dB_t^1 \right\},$$

$$q_T = q_\tau \exp \left\{ \int_\tau^T c_t dt + \int_\tau^T d_t dB_t^1 + \int_\tau^T e_t dB_t^2 \right\},$$

where $q_t = \mu_p(t) - \frac{1}{2}\sigma_p^2(t)$, $b_t = \sigma_p(t)$, $c_t = \mu_q(t) - \frac{1}{2}\sigma_{pq}^2(t) - \frac{1}{2}\sigma_q^2(t)$, $d_t = \sigma_{pq}(t)$, and $e_t = \sigma_q(t)$. It follows that p_T and q_T conditional on time τ follow a bivariate lognormal distribution: $(\log p_T, \log q_T) \sim N(m_1, m_q, s^2, u_q^2, \phi)$, where

$$m_1 = E_\tau[\log p_T] = \log p_\tau + \int_\tau^T a_t dt,$$

$$m_q = E_\tau[\log q_T] = \log q_\tau + \int_\tau^T c_t dt,$$

$$s^2 = \text{Var}_\tau(\log p_T) = \int_\tau^T b_t^2 dt,$$

$$u_q^2 = \text{Var}_\tau(\log q_T) = \int_\tau^T (d_t^2 + e_t^2) dt,$$

$$\begin{aligned}
\phi &= \text{Corr}_\tau(\log p_T, \log q_T) \\
&= (E_\tau[\log p_T \log q_T] - E_\tau[\log p_T]E_\tau[\log q_T]) / (s \cdot u_q) \\
&= \left(\left(\log p_\tau + \int_\tau^T a_i dt \right) \left(\log q_\tau + \int_\tau^T c_i dt \right) \right. \\
&\quad \left. + \int_\tau^T b_i d_i dt - \left(\log p_\tau + \int_\tau^T a_i dt \right) \right. \\
&\quad \left. \cdot \left(\log q_\tau + \int_\tau^T c_i dt \right) \right) / (s \cdot u_q) \\
&= \left(\int_\tau^T b_i d_i dt \right) / (s \cdot u_q).
\end{aligned}$$

Equation (16) is then calculated to give the following function:

$$\begin{aligned}
x_\tau^*(p_T) &= (p_T - r) \exp(m_q + \varphi u_q s (\log p_T - m_1) + 12u_q^2(1 - \varphi^2)) \\
&\quad + r \exp(m_q + 12u_q^2) \\
&\quad - \exp(m_1 + m_q + 12(s^2 + u_q^2) + \varphi s u_q). \quad (17)
\end{aligned}$$

Equation (13) is obtained by substituting the parameters m_1 , m_q , s , u_q , and ϕ into Equation (17). \square

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