Detection of Island Formation and Identification of Causal Factors Under Multiple Line Outages

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***Abstract*—The detection of island formation in power networks is prerequisite for the study of security analysis and control. We develop a combined graph-theoretic-algebraic approach to detect island formation in power system networks under multiple line outages. We construct the approach by gaining insights into the topological impacts of outaged lines on system connectivity from the use of power transfer distribution factor information. We develop a one-to-one relationship between minimal cutsets and a matrix of the generalized line outage distribution factors for multiple line outages. This relationship requires computations on lower order matrices and so is able to provide rapidly essential information. The proposed approach detects the island formation and identifies the subset of outaged lines that is the causal factor. Furthermore, for cases in which the set of outaged lines does not result in system separation, the method has the ability to identify whether a set of candidate line outages separates the system. Consequently, the need for establishing nodal system connectivity is bypassed. We illustrate the capabilities of the proposed approach on two large-scale networks. The proposed approach provides an effective tool for both real-time and offline environments for security analysis and control.**

***Index Terms*—Island formation, Jacobian singularity, line outage distribution factors, minimal cutsets, multiple line outages, power transfer distribution factors.**

I. Introduction

POWER systems are continuously subject to various disturbances such as changes in the loads and the availability of components. Our focus in this paper is on the network topology modifications that separate the system into islands. We study the causality factors of island formation in the presence of multiple line outages and develop a general methodology for its detection and for the identification of the subset of outaged lines causing island formation.

The detection/identification of island formation provides the information needed to be able to deal effectively with the numerous complications that arise. These complications all stem from the singularity of the Jacobian matrix in the Newton power flow [1]–[3]. Consequently, the power flow cannot be used without the introduction of a modified Jacobian matrix. Furthermore, the impacts of the Jacobian matrix singularity propagate through all the applications programs that use the Newton power flow, such as state estimation and various network analysis tools. These complications prevent the use of such tools in standard form and require their application to the connected subnetworks that are formed. Moreover, separation into two or more islands requires the deployment of different control strategies to ensure system security. For offline static security analysis studies, involving the analysis of numerous *what if* cases, the line outages that lead to island formation are regarded as "most problematic" [4], [5]. Indeed, situations with multiple line outages may require extensive corrective control efforts, ranging from redispatch to load shedding—a last resort. The impacts of such line outages are even more pronounced when stability aspects are included [6].

For both real-time as well as offline applications, the rapid detection of island formation and the identification of the causal factors are required to deal with the complications cited above. In cases where several lines are outaged and no island formation occurs, additional network analysis is needed to identify which additional line outage(s) result in system separation into islands.

There exist various methods to detect island formation [1], [7]–[15]. We may classify the existing tools into three major categories—linked list approaches, numerical methods, and graph-theoretic schemes. The pioneering works on real-time island detection are based on the use of linked list tables [7], [8]. Different numerical methods to determine system connectivity use a sequence of multiplications of the network node-to-node connectivity matrix [14], *LU* decomposition for detecting Jacobian singularity [1], and eigen-system evaluation of the nodal susceptance matrix of the augmented network [15]. The graph-theoretic schemes include breadth first search [9], [10], path finding approaches [11], node fusion [12], and two-stage processes [13]. The underlying concept in these graph-theoretic schemes is the determination of *paths* between node pairs of the system. The topology-based approaches use the status of each line to determine the system connectivity. The status information simply indicates whether or not a line in the network is connected. When multiple line outages are involved, the topological approaches may not be the most appropriate tools because every change in the topology requires a new application of the topological algorithm starting from "*scratch*." It follows, then, that the analysis of a cascading situation requires the repeated application of the topology-based scheme to each outage condition whenever a sequence of outages is considered. Since the information on any subset of outages is not utilized, the multiple applications of the topology-based scheme may entail computational inefficiencies. While these considerations are critical in online studies, the many offline studies of various *what if* cases may also be impacted by the computational requirements. In this paper, we address the need of the rapid identification of island formation in a computationally efficient way in which we make effective use of the connectivity information of a subset of outaged lines in a larger set of outaged lines containing that subset.

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Specifically, we propose the development of a combined-graph-theoretic-algebraic approach to detect island formation and to identify the causality factors under multiple line outages. The proposed approach is based on the graph-theoretic notion of minimal cutsets and the approximate line flow sensitivities, the so-called power transfer distribution factors (*PTDF*s). The marriage of the purely topological minimal cutset notion—the outages of the elements of the minimal cutset separate the system—with the circuit theory-based *PTDF*s embodying both topology and network parameter information, harnesses effectively this information. We use the *PTDF*s to evaluate the impacts of line outages on the non-outaged lines' flows in terms of the so-called line outage distribution factors (*LODF*s). The *LODF* values provide the fractions of the pre-outage flow on the outaged line that are redistributed to the non-outaged lines in the post-outage network. Since the focus here is on the study of multiple line outages, we generalize the *LODF* concept to construct what we term the generalized *LODF*s (*GLODF*s) for such cases. In the paper, we establish a one-to-one relationship between *GLODF*s and the minimal cutset. The *GLODF* values of a set of outaged lines become undefined if and only if the set of outaged lines constitutes one or more minimal cutsets. We use this relationship to detect island formation. Moreover, we can also identify the elements of the minimal cutsets and which terminal nodes of the minimal cutset elements are located in the same island.

A salient feature of the proposed approach is its low computational requirements as the computations are carried out on matrices whose dimension is the number of outaged lines. These computations take advantage of the structural characteristics of the proposed method whereby a set of k-line outages serves to establish the results of any larger set of outaged lines containing this k-line set as a subset. For concreteness, under the outages of k-lines, not separating the system, the proposed method evaluates the k-line *GLODF* values. For an additional line outage, resulting in a total of $(k+1)$ line outages, the k-line *GLODF* values are used to compute the $(k+1)$ line *GLODF* values. If all the $(k+1)$ line *GLODF* values are defined, then the $(k+1)$ line outages do not result in island formation; otherwise, two or more islands are formed. In this way, we can directly pinpoint the impact of the interactions between the additional line outage and the k-line outages as a causal factor for island formation. For this reason, the proposed method is particularly useful in the analysis of appropriate preventive/corrective control strategies in cases involving the *domino effect* of multiple line outages to effectively mitigate the impacts of such a sequence of outages.

We limit the scope of this paper to the detection of island formation and the identification of minimal cutset elements. This paper has four more sections. In Section II, we present the power system network model and state the graph-theoretic notions we use. Then, in Section III, we use generalized line outage distribution factors for multiple line outages to evaluate their impacts. We provide the theoretical basis from which we derive the proposed scheme. In Section IV, we illustrate the application of the approach to two large networks—the IEEE 118 bus-system and a 2200-bus network derived from Northeast Power Coordinating Council network. For improved reading of this paper, we present the mathematical details in three appendexes. Appendix A provides the notation for this paper. Appendix B reviews briefly the basic distribution factors and presents the derivation of the *GLODF*s. The proofs of the mathematical statements are given in Appendix C.

II. Power System Network

Our focus in this paper is on the topological modifications of the power network due to line outages. Since the interest is in the connectivity information, it suffices to consider only the real power flows in the network. For these purposes, we use the linear network model in our analysis. We briefly review the power system network model and state specific graph-theoretic notions we use in this paper.

We consider a power system consisting of $(N+1)$ buses and L lines. We denote by $\boldsymbol{\mathcal{N}} = \{0, 1, \ldots, N\}$ the set of buses, with the bus 0 being the slack bus and by $\boldsymbol{\mathcal{L}} = \{\ell_1, \ell_2, \ldots, \ell_L\}$ the set of transmission lines and transformers that connect the buses in the set $\boldsymbol{\mathcal{N}}$. We associate with each line $\ell_m \in \boldsymbol{\mathcal{L}}, m = 1, \ldots, L$, the *ordered* pair of nodes (i_m, j_m), with the convention that the direction of the line ℓ_m real power flow is *from* i_m *to* j_m. We denote by $\underline{\boldsymbol{B}}_{\boldsymbol{d}} \triangleq \operatorname{diag}\{b_1, \ldots, b_L\}$, the branch susceptance matrix and use $\underline{\boldsymbol{A}} \in \mathbb{R}^{L \times N}$ [16] the reduced incidence matrix to construct the reduced nodal susceptance matrix $\underline{\boldsymbol{B}}$ by $\underline{\boldsymbol{B}} = \underline{\boldsymbol{A}}^{\boldsymbol{T}} \underline{\boldsymbol{B}}_{\boldsymbol{d}} \underline{\boldsymbol{A}}$, which is symmetric positive definite [17].

As the focus of this paper is on the topology structure of the power system network, we use the DC power flow model [18] in our analysis. The state is the vector of nodal voltage angles $\underline{\boldsymbol{\theta}} = [\theta_1, \ldots, \theta_N]^T$ given by $\underline{\boldsymbol{B}}\underline{\boldsymbol{\theta}} = \underline{\boldsymbol{p}}$. Here, $\underline{\boldsymbol{p}} \triangleq [p_1, \ldots, p_N]^T$ is the vector of net nodal real power injections. The vector of the real power line flows is $\underline{\boldsymbol{f}} = \underline{\boldsymbol{B}}_d \underline{\boldsymbol{A}}\underline{\boldsymbol{\theta}}$. We represent a t *MW* transaction as an injection of t *MW* at node i and a withdrawal of t *MW* at node j. We denote the t *MW* transaction from node i to node j by the *ordered* triplet $w \triangleq \{i, j, t\}$. The impact of the transaction w on line ℓ_k real power flow is given by the *PTDF* $\varphi_{\ell_k}^{w}$, with

$$\Delta f_{\ell_k} = \varphi_{\ell_k}^{w} t. \tag{1}$$

The impact of line $\tilde{\ell}_m$ outage on line ℓ_k real power flow is $\Delta f_{\ell_k}^{(\tilde{\ell}_m)}$, which may be determined in terms of *LODF* $\varsigma_{\ell_k}^{(\tilde{\ell}_m)}$ and the pre-outage real power flow $f_{\tilde{\ell}_m}$ using the relationship

$$\Delta f_{\ell_k}^{(\tilde{\ell}_m)} = \varsigma_{\ell_k}^{(\tilde{\ell}_m)} f_{\tilde{\ell}_m}. \tag{2}$$

Here, $\varsigma_{\ell_k}^{(\tilde{\ell}_m)} = \varphi_{\ell_k}^{w(\tilde{\ell}_m)} (1 - \varphi_{\tilde{\ell}_m}^{w(\tilde{\ell}_m)})^{-1}$ for $\varphi_{\tilde{\ell}_m}^{w(\tilde{\ell}_m)} \neq 1$. We review distribution factors and derive the *GLODF*s for multiple line outages in Appendix B.

We associate the graph $\boldsymbol{\mathcal{G}} = (\boldsymbol{\mathcal{N}}, \boldsymbol{\mathcal{L}})$ with the power system network, with the set of vertices $\boldsymbol{\mathcal{N}}$ and $\boldsymbol{\mathcal{L}}$ the set of edges of the graph $\boldsymbol{\mathcal{G}}$. We use the terms graph and network interchangeably in the remainder of this paper. $\boldsymbol{\mathcal{G}}$ is a connected network if and only if there exists a path between every pair of nodes i and j. Two connected subnetworks $\boldsymbol{\mathcal{G}_a} = (\boldsymbol{\mathcal{N}_a}\boldsymbol{\mathcal{L}_a})$ and $\boldsymbol{\mathcal{G}_b} = (\boldsymbol{\mathcal{N}_b}, \boldsymbol{\mathcal{L}_b})$ of $\boldsymbol{\mathcal{G}}$ are disjoint if $\boldsymbol{\mathcal{N}_a} \bigcap \boldsymbol{\mathcal{N}_b} = \emptyset$, with $\boldsymbol{\mathcal{N}_a}, \boldsymbol{\mathcal{N}_b} \subset \boldsymbol{\mathcal{N}}$ and $\boldsymbol{\mathcal{L}_a}, \boldsymbol{\mathcal{L}_b} \subseteq \boldsymbol{\mathcal{L}}$. We call a subset $\bar{\boldsymbol{\mathcal{L}}} \subset \boldsymbol{\mathcal{L}}$ a cutset if and only if the removal of all the elements in $\bar{\boldsymbol{\mathcal{L}}}$ from $\boldsymbol{\mathcal{L}}$ partitions $\boldsymbol{\mathcal{G}}$ into two disjoint connected subnetworks $\boldsymbol{\mathcal{G}_a} = (\boldsymbol{\mathcal{N}_a}, \boldsymbol{\mathcal{L}_a})$ and $\boldsymbol{\mathcal{G}_b} = (\boldsymbol{\mathcal{N}_b}, \boldsymbol{\mathcal{L}_b})$ with $\boldsymbol{\mathcal{N}_a} \cup \boldsymbol{\mathcal{N}_b} = \boldsymbol{\mathcal{N}}$. We refer to the separated subnetworks as the islanded subnetworks. A cutset $\hat{\boldsymbol{\mathcal{L}}} \subset \boldsymbol{\mathcal{L}}$ is a *minimal cutset* if no proper subset of $\hat{\boldsymbol{\mathcal{L}}}$ is a cutset [12]. We make use of the development of this section in the analysis of island formation.

III. Development of the Proposed Approach

To start out, we focus on island formation. We use the distribution factors to determine the characterization of a minimal cutset of the power network. Consider the single line $\ell_k \in \boldsymbol{\mathcal{L}}$ and assume that $\hat{\boldsymbol{\mathcal{L}}} = \{\ell_k\} = \{(i_k, j_k)\}$ constitutes a minimal cutset of the connected network $\boldsymbol{\mathcal{G}} = (\boldsymbol{\mathcal{N}}, \boldsymbol{\mathcal{L}})$. The line ℓ_k connects the two subnetworks $\boldsymbol{\mathcal{G}}'_{\boldsymbol{a}}$ and $\boldsymbol{\mathcal{G}}'_{\boldsymbol{b}}$ of $\boldsymbol{\mathcal{G}}$ with $i_k \in \boldsymbol{\mathcal{N}}'_{\boldsymbol{a}}$ and $j_k \in \boldsymbol{\mathcal{N}}'_{\boldsymbol{b}}$. We consider *a transaction* between the terminal nodes of ℓ_k, which we denote by $w(\ell_k) = \{i_k, j_k, t\}$. From the definition of the *PTDF*s in Appendix B, it follows that ℓ_k is a minimal cutset if and only if

$$\varphi_{\ell_m}^{w(\ell_k)} = \begin{cases} 1, & \ell_m = \ell_k \\ 0, & \ell_m \neq \ell_k. \end{cases} \tag{3}$$

We infer from (3) that for the singleton minimal cutset $\hat{\boldsymbol{\mathcal{L}}}$, the *PTDF*s are binary valued for injection/withdrawal at the terminal nodes of a single element minimal cutset. In fact, any inter- and intra-subnetwork transaction, $w \triangleq \{i, j, t\}$, in $\boldsymbol{\mathcal{G}}$ for which $\{\ell_k\}$ constitutes a minimal cutset is characterized by

$$\left|\varphi_{\ell_k}^{\{i,j,t\}}\right| = \begin{cases} 1, & i \in \boldsymbol{\mathcal{N}}'_{\boldsymbol{a}}, j \in \boldsymbol{\mathcal{N}}'_{\boldsymbol{b}} \text{ or } i \in \boldsymbol{\mathcal{N}}'_{\boldsymbol{b}}, j \in \boldsymbol{\mathcal{N}}'_{\boldsymbol{a}} \\ 0, & i, j \in \boldsymbol{\mathcal{N}}'_{\boldsymbol{a}} \text{ or } i, j \in \boldsymbol{\mathcal{N}}'_{\boldsymbol{b}}. \end{cases} \tag{4}$$

In words, if the transaction terminal nodes are in the two different subnetworks with the respective node sets $\boldsymbol{\mathcal{N}}'_{\boldsymbol{a}}$ and $\boldsymbol{\mathcal{N}}'_{\boldsymbol{b}}$, then the transaction must flow over the line ℓ_k; else, there is no net flow on the line ℓ_k when the terminal nodes of the transaction are in the same subnetwork node set $\boldsymbol{\mathcal{N}}'_{\boldsymbol{a}}$ or $\boldsymbol{\mathcal{N}}'_{\boldsymbol{b}}$. Note that $\varphi_{\ell_k}^{w(\ell_k)} = 1$ makes $\varsigma_{\ell_m}^{(\ell_k)}$ undefined.

Consider the minimal cutset $\hat{\boldsymbol{\mathcal{L}}}_{(\beta)} = \{\hat{\ell}_1, \ldots, \hat{\ell}_\beta\}$[1] with β lines connecting two disjoint connected subnetworks $\boldsymbol{\mathcal{G}}_{\boldsymbol{a}} = (\boldsymbol{\mathcal{N}}_{\boldsymbol{a}}, \boldsymbol{\mathcal{L}}_{\boldsymbol{a}})$ and $\boldsymbol{\mathcal{G}}_{\boldsymbol{b}} = (\boldsymbol{\mathcal{N}}_{\boldsymbol{b}}, \boldsymbol{\mathcal{L}}_{\boldsymbol{b}})$ of $\boldsymbol{\mathcal{G}}$. We consider the outages of the first $\beta - 1$ elements of $\hat{\boldsymbol{\mathcal{L}}}_{(\beta)}$, resulting in the modified network $\boldsymbol{\mathcal{G}}'' = (\boldsymbol{\mathcal{N}}, \boldsymbol{\mathcal{L}}'')$, with $\boldsymbol{\mathcal{L}}'' = \boldsymbol{\mathcal{L}} \backslash \{\hat{\ell}_1, \ldots, \hat{\ell}_{\beta-1}\}$. Since $\hat{\boldsymbol{\mathcal{L}}}_{(\beta)}$ is a minimal cutset, $\boldsymbol{\mathcal{G}}''$ is connected. Also, $\hat{\ell}_\beta$ is a minimal cutset of $\boldsymbol{\mathcal{G}}''$ connecting the two disjoint subnetworks $\boldsymbol{\mathcal{G}}_{\boldsymbol{a}}$ and $\boldsymbol{\mathcal{G}}_{\boldsymbol{b}}$ of $\boldsymbol{\mathcal{G}}''$. Therefore, (3) states that for this network

$$\left(\varphi_{\ell_k}^{w(\hat{\ell}_\beta)}\right)^{(\hat{\boldsymbol{\mathcal{L}}}_{(\beta-1)})} = \begin{cases} 1, & \ell_k = \hat{\ell}_\beta \\ 0, & \ell_k \neq \hat{\ell}_\beta \end{cases}, \quad \ell_k \notin \hat{\boldsymbol{\mathcal{L}}}_{(\beta-1)}. \tag{5}$$

Since the terminal nodes of each minimal cutset element are in the two different subnetworks, the discussion after (4) implies

$$\left|\left(\varphi_{\hat{\ell}_\beta}^{w(\hat{\ell}_k)}\right)^{(\hat{\boldsymbol{\mathcal{L}}}_{(\beta-1)})}\right| = 1, \quad \hat{\ell}_k \in \hat{\boldsymbol{\mathcal{L}}}_{(\beta)}. \tag{6}$$

Thus, once all but one elements of $\hat{\boldsymbol{\mathcal{L}}}_{(\beta)}$ are outaged, the pre-outage real power flow in each outaged line has to flow over the minimal cutset element that is not outaged. We can also show the case of all but two element outages of $\hat{\boldsymbol{\mathcal{L}}}_{(\beta)}$

$$\left|\left(\varphi_{\hat{\ell}_\beta}^{w(\hat{\ell}_k)}\right)^{(\hat{\boldsymbol{\mathcal{L}}}_{(\beta-2)})}\right| + \left|\left(\varphi_{\hat{\ell}_{\beta-1}}^{w(\hat{\ell}_k)}\right)^{(\hat{\boldsymbol{\mathcal{L}}}_{(\beta-2)})}\right| = 1, \ \hat{\ell}_k \in \hat{\boldsymbol{\mathcal{L}}}_{(\beta)}. \tag{7}$$

[1]While $\hat{\boldsymbol{\mathcal{L}}}_{(\beta)}$ is not an ordered set, we reorder the elements from 1 to β so as to allow the use of simple notation.

Then, by induction, we can rigorously establish that

$$\sum_{\hat{\ell}_k \in \hat{\boldsymbol{\mathcal{L}}}_{(\beta)}} \left|\varphi_{\hat{\ell}_k}^{w(\hat{\ell}_m)}\right| = 1, \quad \hat{\ell}_m \in \hat{\boldsymbol{\mathcal{L}}}_{(\beta)}. \tag{8}$$

We use (8) to prove the following.

Theorem 1*:* Let $\hat{\boldsymbol{\mathcal{G}}} = (\boldsymbol{\mathcal{N}}, \boldsymbol{\mathcal{L}})$ be a connected power system network. The minimal cutset $\hat{\boldsymbol{\mathcal{L}}}_{(\beta)} = \{\hat{\ell}_1, \ldots, \hat{\ell}_\beta\}$ partitions $\boldsymbol{\mathcal{G}}$ into two subnetworks $\boldsymbol{\mathcal{G}}_a = (\boldsymbol{\mathcal{N}}_a, \boldsymbol{\mathcal{L}}_a)$ and $\boldsymbol{\mathcal{G}}_b = (\boldsymbol{\mathcal{N}}_b, \boldsymbol{\mathcal{L}}_b)$. Each line $\hat{\ell}_k = (\hat{i}_k, \hat{j}_k) \in \hat{\boldsymbol{\mathcal{L}}}$ has $\hat{i}_k \in \boldsymbol{\mathcal{N}}_a$ and $\hat{j}_k \in \boldsymbol{\mathcal{N}}_b$. Then

$$(i) \quad \varphi_{\hat{\ell}_k}^{w(\hat{\ell}_m)} > 0, \quad \hat{\ell}_m, \hat{\ell}_k \in \hat{\boldsymbol{\mathcal{L}}} \tag{9}$$

$$(ii) \quad \sum_{\hat{\ell}_k \in \hat{\boldsymbol{\mathcal{L}}}} \varphi_{\hat{\ell}_k}^{\{i,j,t\}} = \begin{cases} 1, & i \in \boldsymbol{\mathcal{N}}_a, j \in \boldsymbol{\mathcal{N}}_b \\ -1, & i \in \boldsymbol{\mathcal{N}}_b, j \in \boldsymbol{\mathcal{N}}_a \end{cases} \tag{10}$$

$$(iii) \sum_{\hat{\ell}_k \in \hat{\boldsymbol{\mathcal{L}}}} \varphi_{\hat{\ell}_k}^{\{i,j,t\}} = 0, \ i, j \in \boldsymbol{\mathcal{N}}_a \text{ or } \quad i, j \in \boldsymbol{\mathcal{N}}_b. \tag{11}$$

■

We prove Theorem 1 in Appendix C. This theorem provides the necessary conditions of minimal cutsets. We also examine the physical intuitions behind these conditions. The first part is a generalized restatement of (8). The second part states any transaction between $\boldsymbol{\mathcal{N}}_a$ and $\boldsymbol{\mathcal{N}}_b$ results in net power flows on the minimal cutset elements; the algebraic sum of the minimal cutset flows equals to the transaction amount. The last part states that any transaction between nodes of $\boldsymbol{\mathcal{N}}_a(\boldsymbol{\mathcal{N}}_b)$ results in 0 net power flow across the minimal cutset.

We next consider a set of α outaged lines denoted by $\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)} = \{\tilde{\ell}_1, \tilde{\ell}_2, \ldots, \tilde{\ell}_\alpha\}$, which need not be a cutset. To determine whether $\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}$ contains one or more minimal cutsets, we make use of the *GLODF*s. For line $\ell_k \notin \tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}$ with all the α lines in $\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}$ outaged, the vector $\underline{\xi}_{\ell_k}^{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}} \in \mathbb{R}^\alpha$ is given

$$\underline{\boldsymbol{H}}_\alpha^T \underline{\xi}_{\ell_k}^{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}} = \left[\varphi_{\ell_k}^{w(\tilde{\ell}_1)}, \varphi_{\ell_k}^{w(\tilde{\ell}_2)}, \ldots, \varphi_{\ell_k}^{w(\tilde{\ell}_\alpha)}\right]^T \tag{12}$$

where

$$\underline{\boldsymbol{H}}_\alpha \triangleq \left[\underline{\boldsymbol{I}}^\alpha - \underline{\boldsymbol{\Phi}}_{\tilde{\mathcal{L}}_{(\alpha)}}\right] = \begin{bmatrix} 1 - \varphi_{\tilde{\ell}_1}^{w(\tilde{\ell}_1)} & \cdots & -\varphi_{\tilde{\ell}_1}^{w(\tilde{\ell}_\alpha)} \\ \vdots & \ddots & \vdots \\ -\varphi_{\tilde{\ell}_\alpha}^{w(\tilde{\ell}_1)} & \cdots & 1 - \varphi_{\tilde{\ell}_\alpha}^{w(\tilde{\ell}_\alpha)} \end{bmatrix}. \tag{13}$$

As long as $\underline{\boldsymbol{H}}_\alpha$ is nonsingular, (12) uniquely determines the LODFs for a line $\ell_k \notin \tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}$ of the outaged lines in $\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}$. We derive an important relationship between the singularity of $\underline{\boldsymbol{H}}_\alpha$ and the existence of minimal cutsets in $\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}$ in the following.

Theorem 2*:* Let $\boldsymbol{\mathcal{G}} = (\boldsymbol{\mathcal{N}}, \boldsymbol{\mathcal{L}})$ be a connected power system network. For a set of α outaged lines $\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)} = \{\tilde{\ell}_1, \tilde{\ell}_2, \ldots, \tilde{\ell}_\alpha\}$

$$\underline{\boldsymbol{H}}_\alpha \text{ is singular} \Leftrightarrow \begin{array}{l} \tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)} \text{ contains one or} \\ \text{more minimal cutsets.} \end{array}$$

■

We provide the proof in Appendix C. The singularity of $\underline{\boldsymbol{H}}_\alpha$ is equivalent to the existence of a $\underline{\boldsymbol{u}} \in \mathbb{R}^\alpha$, such that $\underline{\boldsymbol{u}}^T \underline{\boldsymbol{H}}_\alpha = (\underline{\boldsymbol{0}}^\alpha)^T$. By definition, $\underline{\boldsymbol{u}}$ is a (left) eigenvector corresponding to a zero eigenvalue of $\underline{\boldsymbol{H}}_\alpha$, i.e., $\underline{\boldsymbol{u}}$ is an element of the left nullspace $\boldsymbol{\mathcal{R}}(\underline{\boldsymbol{H}}_\alpha^T)$, where

$$\boldsymbol{\mathcal{R}}\left(\underline{\boldsymbol{H}}_\alpha^T\right) = \left\{\boldsymbol{x}: \boldsymbol{x}^T \underline{\boldsymbol{H}}_\alpha^T = \underline{\boldsymbol{0}}^T, \boldsymbol{x} \neq \underline{\boldsymbol{0}}\right\}. \tag{14}$$

If $\underline{\boldsymbol{H}}_\alpha$ has $\rho(\alpha)$ zero eigenvalues, we can use Theorem 2 and show that each zero eigenvalue has a unique (up to a scaling factor) eigenvector $\underline{\boldsymbol{u}}^i$ and is distinct,[2] $i = 1, \ldots, \rho(\alpha)$. Consequently, $\{\underline{\boldsymbol{u}}^i: i = 1, \ldots, \rho(\alpha)\}$ forms a basis for $\boldsymbol{\mathcal{R}}(\underline{\boldsymbol{H}}_\alpha^T)$.

Theorem 2 provides graph-theoretic insights into minimal cutsets. In the realm of the rank of $\underline{\boldsymbol{H}}_\alpha$, we establish that the $\operatorname{rank}(\underline{\boldsymbol{H}}_\alpha) = \alpha - \rho(\alpha)$, and so, there is equivalence of the existence of $\rho(\alpha)$ minimal cutsets in $\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}$ with that rank. We denote each such minimal cutset by $\hat{\boldsymbol{\mathcal{L}}}^i, i = 1, \ldots, \rho(\alpha)$. By definition, $\hat{\boldsymbol{\mathcal{L}}}^i$ partitions $\boldsymbol{\mathcal{G}} = (\boldsymbol{\mathcal{N}}, \boldsymbol{\mathcal{L}})$ into the disjoint subnetworks $\boldsymbol{\mathcal{G}}_a^i = (\boldsymbol{\mathcal{N}}_a^i, \boldsymbol{\mathcal{L}}_a^i)$ and $\boldsymbol{\mathcal{G}}_b^i = (\boldsymbol{\mathcal{N}}_b^i, \boldsymbol{\mathcal{L}}_b^i)$.

We next find a basis for $\rho\ (\alpha)$ dimensional $\boldsymbol{\mathcal{R}}(\underline{\boldsymbol{H}}_\alpha^T)$ by the *rank revealing QR* (*RRQR*) factorization [20], [21] of $\underline{\boldsymbol{H}}_\alpha^T$

$$\underline{\boldsymbol{H}}_\alpha^T \underline{\boldsymbol{P}} = \underline{\boldsymbol{Q}}\underline{\boldsymbol{R}} = \left[\underline{\boldsymbol{Q}}_{\omega(\alpha)} \underline{\boldsymbol{Q}}_{\rho(\alpha)}\right] \times \begin{bmatrix} \underline{\boldsymbol{R}}_{\omega(\alpha),\omega(\alpha)} & \underline{\boldsymbol{R}}_{\omega(\alpha),\rho(\alpha)} \\ \underline{\boldsymbol{0}} & \underline{\boldsymbol{R}}_{\rho(\alpha),\rho(\alpha)} \end{bmatrix} \tag{15}$$

where $\omega(\alpha) \triangleq \alpha - \rho(\alpha)$. Here, $\underline{\boldsymbol{R}}_{\rho(\alpha),\rho(\alpha)} \in \mathbb{R}^{\rho(\alpha),\rho(\alpha)}$ is the submatrix with the 0 diagonal elements. The set of the columns of the $\underline{\boldsymbol{Q}}_{\rho(\alpha)}$ forms an orthonormal basis for the nullspace of $\boldsymbol{\mathcal{N}}(\underline{\boldsymbol{H}}_\alpha^T)$ [20]. We next transform this basis so as to identify the elements in each minimal cutset using the transformation matrix $\underline{\boldsymbol{T}} \in \mathbb{R}^{\rho(\alpha),\rho(\alpha)}$

$$\left[\underline{\boldsymbol{v}}_1 \vdots \underline{\boldsymbol{v}}_2 \vdots \cdots \vdots \underline{\boldsymbol{v}}_{\rho(\alpha)}\right] \underline{\boldsymbol{T}} = \underline{\boldsymbol{Q}}_{\rho(\alpha)}. \tag{16}$$

The construction of $\underline{\boldsymbol{T}}$ is detailed in Appendix C. It follows that $\{\underline{\boldsymbol{v}}^i: i = 1, \ldots, \rho(\alpha)\}$ *is a basis for* $\boldsymbol{\mathcal{R}}(\underline{\boldsymbol{H}}_\alpha^T)$ with

$$\left|v_r^i\right| = \begin{cases} 1, & \tilde{\ell}_r \in \hat{\boldsymbol{\mathcal{L}}}^i \\ 0, & \tilde{\ell}_r \notin \hat{\boldsymbol{\mathcal{L}}}^i. \end{cases} \tag{17}$$

Each $\underline{\boldsymbol{v}}^i, i = 1, \ldots, \rho(\alpha)$ identifies the elements of the corresponding minimal cutset, i.e., the subset of lines without which system separates into two islands. For two lines $\tilde{\ell}_r, \tilde{\ell}_s \in \hat{\boldsymbol{\mathcal{L}}}^i$, we check $\delta_{r,s} = \operatorname{sign}\{v_r^i v_s^i\}$. If $\delta_{r,s} > 0$, then the from (to) terminal nodes of $\tilde{i}_r$ and $\tilde{i}_s$ ($\tilde{j}_r$ and $\tilde{j}_s$) are in the same island. For $\delta_{r,s} < 0$, however, $\tilde{i}_r$ and $\tilde{i}_s$ ($\tilde{j}_r$ and $\tilde{j}_s$) are in the two separate islands.

The analysis of $\underline{\boldsymbol{H}}_\alpha$ provides insights into the formation of islands under multiple line outages. For detection of island formation, we use Gaussian elimination, and for identification of the elements of each minimal cutset, we obtain the *RRQR* factors of $\underline{\boldsymbol{H}}_\alpha^T$. In the Gaussian elimination of $\underline{\boldsymbol{H}}_\alpha$, a zero diagonal element at some elimination step k corresponds to

$$h_{i,i}^{(k)} = 1 - \left(\varphi_{\tilde{\ell}_i}^{w(\tilde{\ell}_i)}\right)^{(\tilde{\boldsymbol{\mathcal{L}}}_{(k-1)})} = 0. \tag{18}$$

[2]In fact, the Jordan canonical form of $\underline{\boldsymbol{H}}_\alpha$ has Jordan submatrices that correspond to the 0 eigenvalues with "order unity" [19].

TABLE I
PTDF VALUES FOR A SUBSET OF OUTAGED LINES

line	node pair	$w(\tilde{\ell}_1)$	$w(\tilde{\ell}_2)$	$w(\tilde{\ell}_3)$	$w(\tilde{\ell}_4)$	$w(\tilde{\ell}_5)$	$w(\tilde{\ell}_6)$	$w(\tilde{\ell}_7)$
$\tilde{\ell}_1$	(6,7)	0.8571	-0.0145	-0.0195	-0.0103	-0.0005	-0.001	0.0002
$\tilde{\ell}_2$	(33,37)	-0.0021	0.6045	0.2075	-0.0926	0.0113	0.0226	-0.0041
$\tilde{\ell}_3$	(19,34)	-0.0016	0.1193	0.3115	-0.0913	0.0126	0.0251	-0.0045
$\tilde{\ell}_4$	(38,30)	-0.0040	-0.2435	-0.4177	0.7644	-0.0471	-0.0938	0.0170
$\tilde{\ell}_5$	(23,24)	-0.0002	0.0327	0.0633	-0.0517	0.9290	-0.1414	0.0256
$\tilde{\ell}_6$	(24,72)	-0.0001	0.0163	0.0316	-0.0258	-0.0355	0.6911	0.0559
$\tilde{\ell}_7$	(70,71)	0.0001	-0.0163	-0.0316	0.0258	0.0355	0.3089	0.9441

By Theorem 2 then, there exists at least one minimal cutset contained in $\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}$, and therefore, the outages of the elements of $\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}$ is responsible for the formation of two or more islands. We stop the Gaussian elimination process and proceed with the *RRQR* factorization of $\underline{\boldsymbol{H}}_\alpha^T$. The relations in (15) and (16) establish the number of minimal cutsets and the identity of the elements of each minimal cutset. Moreover, the sign of each pair of $\delta_{r,s}$ in each minimal cutset provides the location of the terminal nodes with respect to the formed islands.

We determine an analytic bound for the total number of multiplication/division operations required in the detection of island formation. For the outage of a set of k-lines, the construction of $\underline{\boldsymbol{H}}_k$ requires no such operations. We note that the dimension of $\underline{\boldsymbol{H}}_k$ is considerably smaller than that of the topological arrays $\underline{\boldsymbol{B}}$ and $\underline{\boldsymbol{A}}$. For this single set of k-line outages, the number of multiplication/division operations in the Gaussian elimination is $O(k^3)$ [23]. When additional line outages are considered, the factors of $\underline{\boldsymbol{H}}_k$ are used. For a single additional line, the number of multiplication/division operations is $O(k^2)$. For, say, q additional line outages, the number of multiplication/division operations is $O(qk^2)$. Note that these bounds represent "worst-case conditions" since no computations are performed once a 0 diagonal element is detected that may be done by inspection.

IV. Applications

The implementation of the proposed approach is straightforward. We illustrate the application of the proposed approach to two different networks—the IEEE 118-bus system and a 2200-bus portion of the large-scale Northeast Power Coordinating Council (NPCC) network. In the connected IEEE 118-bus system, we select a subset of seven of the 194 lines and study the impacts of the outages of this subset. The line definitions and the *PTDF*s are shown in Table I.

For $\hat{\boldsymbol{\mathcal{L}}}_{(7)} = \{\tilde{\ell}_1, \tilde{\ell}_2, \ldots, \tilde{\ell}_7\}$, we compute $\underline{\boldsymbol{H}}_7$ given in (B9) and perform the Gaussian elimination, which produces a zero diagonal element at step 5. The *RRQR* factorization for this matrix determines (19), shown at the bottom of the next page, where the two 0 diagonal elements indicate that $\rho(7) = 2$. The corresponding $\underline{\boldsymbol{Q}}_{\rho(7)}$ given by (15) is in (20) at the bottom of the next page. The two vectors in $\mathbb{R}^7$ given in (20) span $\boldsymbol{\mathcal{R}}(\underline{\boldsymbol{H}}_7^T)$. We construct the transformation matrix

$$\underline{\boldsymbol{T}} = \begin{bmatrix} -0.4989 & -0.0319 \\ 0.0451 & -0.7056 \end{bmatrix} \tag{21}$$

TABLE II
IEEE 118-BUS SYSTEM: $\tilde{\mathcal{L}}_{(7)}$ MINIMAL CUTSET INFORMATION

$\hat{\mathcal{L}}^i$	nodes belonging to $\mathcal{N}_a^i$	nodes belonging to $\mathcal{N}_b^i$
$\{\tilde{\ell}_2, \tilde{\ell}_3, \tilde{\ell}_4, \tilde{\ell}_5\}$	19, 23, 30, 33	24, 34, 37, 38
$\{\tilde{\ell}_6, \tilde{\ell}_7\}$	24, 70	71, 72

TABLE III
OUTAGED LINES AND LINE DEFINITIONS

scheduled maintenance	$\tilde{\ell}_1=(3108,3118)$, $\tilde{\ell}_2=(1796,2929)$, $\tilde{\ell}_3=(1790,2694)$, $\tilde{\ell}_4=(2727,2764)$, $\tilde{\ell}_5=(2718,2721)$	$\tilde{\mathcal{L}}_{(9)}$
switched outages	$\tilde{\ell}_6=(486,2927)$, $\tilde{\ell}_7=(3106,3113)$, $\tilde{\ell}_8=(2716,2765)$, $\tilde{\ell}_9=(2718,2769)$	
unplanned outages	$\tilde{\ell}_{10}=(90,2693)$, $\tilde{\ell}_{11}=(2925,2926)$, $\tilde{\ell}_{12}=(2691,2693)$, $\tilde{\ell}_{13}=(2693,2694)$, $\tilde{\ell}_{14}=(2926,2927)$, $\tilde{\ell}_{15}=(2924,2926)$	

and obtain the transformed basis vector

$$[\underline{\boldsymbol{v}}_1 \vdots \underline{\boldsymbol{v}}_2] = \begin{bmatrix} 0 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}^T. \tag{22}$$

It follows that the zero diagonal element in the Gaussian elimination implies that the system separates into two or more islands when all the seven lines are outaged. However, since $\rho(7) = 2$, we established that there exist two minimal cutsets in the set of $\tilde{\mathcal{L}}_{(7)}$ outaged lines. The components of $\underline{\boldsymbol{v}}_1$ and $\underline{\boldsymbol{v}}_2$ allow us to identify the members of the two minimal cutsets as shown in Table II. We use the relative sign difference of the elements of $\underline{\boldsymbol{v}}_1(\underline{\boldsymbol{v}}_2)$ to determine the nodes of the subnetworks to which the terminal nodes of the lines of in each minimal cutset belong.

Thus, the Gaussian elimination and the *RRQR* factorization provide comprehensive information on the impacts of the outages of the seven lines for the IEEE 118-bus system case.

Next we examine the network with 2200 buses and 2847 lines to evaluate the effects of the scheduled maintenance, switched outages, and unplanned outages. The set of outaged lines is given in Table III. We denote the set of outaged lines corresponding to the scheduled maintenance and the switching actions by $\tilde{\mathcal{L}}_{(9)}$ and consider the network with the elements of $\tilde{\mathcal{L}}_{(9)}$ outaged. We analyze the impacts of the six unplanned line outages listed in Table III on the system connectivity for this network in terms of the so-called updated *PTDF*s. For this purpose, we construct (23), shown at the bottom of the page. The zero diagonal element in (23) implies the singularity of $(\underline{\boldsymbol{H}}_6)^{(\tilde{\mathcal{L}}_{(9)})}$. Hence, in this case, we detect island formation simply by inspection.

For the identification of the minimal cutset elements, we consider all the outaged lines, denoted by the set $\tilde{\mathcal{L}}_{(15)}$, and construct $\underline{\boldsymbol{H}}_{15}$. The *RRQR* factorization results in three zero diagonal elements in $\underline{\boldsymbol{R}}$, so that $\rho(15) = 3$. We evaluate the three basis vectors $[\underline{\boldsymbol{v}}_1 \vdots \underline{\boldsymbol{v}}_2 \vdots \underline{\boldsymbol{v}}_3]$ spanning $\mathcal{R}(\underline{\boldsymbol{H}}_{15}^T)$ using $\underline{\boldsymbol{Q}}_{\rho(15)}$ in (24), shown at the bottom of the next page. The value 3 of $\rho(15)$ indicates the existence of three minimal cutsets in $\tilde{\mathcal{L}}_{(15)}$. The components of

$$\underline{\boldsymbol{R}} = \begin{bmatrix} -0.8354 & 0.0707 & 0.0022 & -0.0061 & -0.1644 & -0.0004 & -0.0079 \\ 0 & -0.4761 & -0.0043 & -0.0086 & -0.1939 & 0.0007 & -0.0117 \\ 0 & 0 & 0.4698 & 0.0010 & 0.0862 & -0.0850 & 0.0786 \\ 0 & 0 & 0 & 0.1425 & 0.0000 & 0.0000 & 0.0000 \\ 0 & 0 & 0 & 0 & -0.0666 & 0.0000 & -0.0607 \\ 0 & 0 & 0 & 0 & 0 & 0.0000 & 0.0000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0000 \end{bmatrix} \tag{19}$$

$$\underline{\boldsymbol{Q}}_{\rho(7)} = \begin{bmatrix} 0.0000 & -0.0319 & -0.0319 & 0.0319 & -0.0319 & -0.7056 & -0.7056 \\ 0.0000 & -0.4989 & -0.4989 & 0.4989 & -0.4989 & 0.0451 & 0.0451 \end{bmatrix}^T \tag{20}$$

$$(\underline{\boldsymbol{H}}_6)^{(\tilde{\mathcal{L}}_{(9)})} = \begin{bmatrix} 0.5349 & 0.0000 & -0.4222 & 0.1127 & 0.0000 & 0.0000 \\ 0.0000 & 0.1556 & 0.0000 & 0.0000 & 0.0000 & -0.1556 \\ -0.1146 & 0.0000 & 0.3011 & 0.1865 & 0.0000 & 0.0000 \\ 0.0466 & 0.0000 & 0.2841 & 0.3307 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.1950 & 0.0000 & 0.0000 & 0.0000 & 0.1950 \end{bmatrix} \tag{23}$$

TABLE IV
LARGE-SCALE NETWORK: $\tilde{\mathcal{L}}_{(15)}$ MINIMAL CUTSET INFORMATION

$\hat{\mathcal{L}}^i$	nodes belonging to $\mathcal{N}_a^i$	nodes belonging to $\mathcal{N}_b^i$
$\{\tilde{\ell}_3,\tilde{\ell}_4,\tilde{\ell}_5,\tilde{\ell}_8,\tilde{\ell}_9,\tilde{\ell}_{10},\tilde{\ell}_{12},\tilde{\ell}_{13}\}$	1790, 2764, 2721, 2765, 2769, 2693	2694, 2727, 2718, 2716, 90, 2691
$\{\tilde{\ell}_6,\tilde{\ell}_{14}\}$	486, 2926	2927
$\{\tilde{\ell}_{11},\tilde{\ell}_{14},\tilde{\ell}_{15}\}$	2925, 2927, 2924	2926

$\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2$ and $\underline{\boldsymbol{v}}_3$ allow us to identify the members of these three minimal cutsets and the corresponding subnetworks associated with the terminal nodes of each minimal cutset. The identification results are summarized in Table IV.

The proposed approach is flexible and does not require the updated *PTDF*s. In fact, to study system connectivity, we first investigate the impacts of the outages of the elements of $\tilde{\mathcal{L}}_{(9)}$. We construct the corresponding $\underline{\boldsymbol{H}}_9$ and perform the Gaussian elimination on $\underline{\boldsymbol{H}}_9$. We determine that the outages of the lines in $\tilde{\mathcal{L}}_{(9)}$ do not form islands. We next focus on the impacts of the unplanned outages of the lines using the sequence given in Table III. For the first outage in this sequence, we augment $\underline{\boldsymbol{H}}_9$ by adding a row and a column to form $\underline{\boldsymbol{H}}_{10}$. We perform the Gaussian elimination on $\underline{\boldsymbol{H}}_{10}$ using the factors of $\underline{\boldsymbol{H}}_9$. Since no islanding results, for the second unplanned outage, we again augment $\underline{\boldsymbol{H}}_{10}$ by adding a row and a column and perform the Gaussian elimination of the resulting $\underline{\boldsymbol{H}}_{11}$ using the factors of $\underline{\boldsymbol{H}}_{10}$. This process continues until either we detect a zero diagonal element in the Gaussian elimination of each augmented matrix in the sequence or we complete the Gaussian elimination of $\underline{\boldsymbol{H}}_{15}$. We can similarly analyze the impacts of any subset of additional outages by using the factors of the unaugmented matrix in the Gaussian elimination step of the augmented matrix.

V. SUMMARY

We develop a combined graph-theoretic-algebraic approach to detect island formation in power system networks under multiple line outages. Rather than rely on the need to conduct *pathfinding* approaches, the proposed approach uses some of the characteristics of minimal cutsets in power system networks. The application of linear algebraic notions allows us the detection of island formation and also identification of the causal factors. We construct the proposed approach making detailed use of the minimal cutset properties. The new method is very useful in both online and offline environments so as to effectively deal with the many complications that arise from island formation. When the outages of multiple lines result in the formation of two or more islands, the method is able to identify which outaged lines cause the system separation. In cases where several lines are outaged and no island formation occurs, the method can identify whether a set of candidate line outages separates the system into islands. Such identification provides the information needed for the deployment of appropriate tools for security analysis and control. A salient characteristic of the proposed approach is the low computing requirements. The extension of the work to the determination of all the nodes of the formed islands will be reported in a future paper.

APPENDIX A
NOTATION

$\mathcal{N} \triangleq \{0,1,2,\ldots,N\}$	Set of $N+1$ buses with the slack bus at node 0.
$\ell_m \triangleq (i_m, j_m)$	Line ℓ_m joining nodes i_m, j_m with the ordered bus pair (i_m, j_m), denoting the from and to buses, respectively.
$\mathcal{L} \triangleq \{\ell_1,\ldots,\ell_L\}$	Set of network transmission lines.
$\mathcal{G} = (\mathcal{N},\mathcal{L})$	Power system network with $\mathcal{N}$ and $\mathcal{L}$.
$\tilde{\mathcal{L}}_{(\alpha)} \triangleq \{\tilde{\ell}_i : \tilde{\ell}_i \in \mathcal{L}\}$	Subset of outaged lines with $\lvert\tilde{\mathcal{L}}_{(\alpha)}\rvert = \alpha$.
$\hat{\mathcal{L}}_{(\beta)} \triangleq \{\hat{\ell}_j : \hat{\ell}_j \in \mathcal{L}\}$	Subset of transmission lines constituting a minimal cutset with $\lvert\hat{\mathcal{L}}_{(\beta)}\rvert = \beta$.

$$[\underline{\boldsymbol{v}}_1 \vdots \underline{\boldsymbol{v}}_2 \vdots \underline{\boldsymbol{v}}_3] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \underline{\boldsymbol{Q}}_{\rho(15)} = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ -0.0440 & 0.3505 & -0.0134 \\ 0.0440 & -0.3505 & 0.0134 \\ 0.0440 & -0.3505 & 0.0134 \\ 0.0595 & 0.0370 & 0.7714 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.0440 & -0.3505 & 0.0134 \\ 0.0440 & -0.3505 & 0.0134 \\ 0.0440 & -0.3505 & 0.0134 \\ 0.5909 & 0.0823 & 0.2097 \\ 0.0440 & -0.3505 & 0.0134 \\ -0.0440 & 0.3505 & -0.0134 \\ -0.5314 & -0.0452 & 0.5617 \\ 0.5909 & 0.0823 & 0.2097 \end{bmatrix} \tag{24}$$

$w \triangleq \{i, j, t\}$	Basic transaction of t *MW* from the node i to the node j.
$\varphi^{w}_{\ell_k}$	*PTDF* of a line ℓ_k due to $w = \{i, j, t\}$.
$(\vartheta)^{(\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)})}$	Value of the variable ϑ in the modified network obtained with all the elements of $\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}$ outaged.
$\varsigma^{(\ell_m)}_{\ell_k}$	*LODF* of a line ℓ_k due to the line ℓ_m outage.
$\underline{\xi}^{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}}_{\ell_k}$	Vector of *GLODF*s of the line ℓ_k due to the outages of the line elements of $\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}$.

Appendix B
Review of the Basic Distribution Factors

We provide a brief review of key aspects of distribution factors: injection shift factors (*ISF*s), *PTDF*, and *LODF*. For a more detailed treatment, the reader is referred to [24] and [25]. In this section, we also develop a generalized expression for *LODF*s under multiple line outages.

For the power system model given in [24] and [25], the *ISF* $\psi^{i}_{\ell_k}$ of a line $\ell_k \in \boldsymbol{\mathcal{L}}$ is the approximate sensitivity of the line ℓ_k real power flow f_{ℓ_k} with respect to the injection p_i at some node $i \in \boldsymbol{\mathcal{N}}$ and a corresponding withdrawal at the slack bus. We use $\psi^{i}_{\ell_k}$ to construct the *ISF* matrix $\underline{\boldsymbol{\Psi}} \in \mathbb{R}^{LxN}$. In fact, we can state $\underline{\boldsymbol{\Psi}}$ analytically as [24]

$$\underline{\boldsymbol{\Psi}} \triangleq \underline{\boldsymbol{B_d}}\underline{\boldsymbol{A}}\underline{\boldsymbol{B}}^{-1}. \tag{B1}$$

Note that each $\psi^{i}_{\ell_k}, \ell_k \in \boldsymbol{\mathcal{L}}$ has flow direction information on the line $\ell_k = (i_k, j_k)$ definition. A positive (negative) $\psi^{i}_{\ell_k}$ value indicates that the injection at node i and withdrawal at node 0 results in the flows from i_k to j_k (j_k to i_k).

The *PTDF* $\varphi^{w}_{\ell_k}$ is the approximate sensitivity of the real power flow f_{ℓ_k} on line ℓ_k with respect to a change in the transaction amount t for the transaction $w \triangleq \{i, j, t\}$ is [25]

$$\varphi^{w}_{\ell_k} \triangleq \psi^{i}_{\ell_k} - \psi^{j}_{\ell_k}. \tag{B2}$$

This definition ensures that $|\varphi^{w}_{\ell_k}| \leqslant 1$ is as shown in [25].

We next construct the *PTDF* matrix for the entire set of $\boldsymbol{\mathcal{L}}$

$$\underline{\boldsymbol{\Phi}}_{\boldsymbol{\mathcal{L}}} = \underline{\boldsymbol{\Psi}}\underline{\boldsymbol{A}}^{\boldsymbol{T}} = \underline{\boldsymbol{B_d}}\underline{\boldsymbol{A}}\left(\underline{\boldsymbol{A}}^{\boldsymbol{T}}\underline{\boldsymbol{B_d}}\underline{\boldsymbol{A}}\right)^{-1}\underline{\boldsymbol{A}}^{\boldsymbol{T}} \tag{B3}$$

is an $L \times L$ matrix with the element $\varphi^{w(\ell_k)}_{\ell_m}$ in row m and column k. We use the notation $w(\ell_k) \triangleq \{i_k, j_k, t\}$, since $\varphi^{w(\ell_k)}_{\ell_m}$ is defined for the transaction between the terminal node pairs $\{i_k, j_k\}$ of ℓ_k as opposed to $\varphi^{w}_{\ell_k}$, where the transaction is defined between any node pair $\{i, j\}$. We can show that $\underline{\boldsymbol{A}}(\underline{\boldsymbol{A}}^{\boldsymbol{T}}\underline{\boldsymbol{B_d}}\underline{\boldsymbol{A}})^{-1}\underline{\boldsymbol{A}}^{\boldsymbol{T}}$ is symmetric positive definite. Since the diagonal elements of the diagonal matrix $\underline{\boldsymbol{B_d}}$ are positive, then $\underline{\boldsymbol{\Phi}}_{\boldsymbol{\mathcal{L}}}$ is structurally symmetric with

$$\begin{aligned} &1 \geqslant \varphi^{w(\ell_r)}_{\ell_r} > 0, \quad \ell_r \in \boldsymbol{\mathcal{L}} \\ &\varphi^{w(\ell_p)}_{\ell_r}\varphi^{w(\ell_r)}_{\ell_p} \geqslant 0, \quad \ell_p, \ell_r \in \boldsymbol{\mathcal{L}}. \end{aligned} \tag{B4}$$

Whenever a change in the network topology occurs, the *PTDF*s of the modified network must be determined. The *LODF*, $\varsigma^{(\ell_m)}_{\ell_k}$, is defined as the portion of the pre-outage real power flow on line $\tilde{\ell}_m$ that is redistributed to line ℓ_k and is

$$\varsigma^{(\tilde{\ell}_m)}_{\ell_k} = \frac{\Delta f^{(\tilde{\ell}_m)}_{\ell_k}}{f_{\tilde{\ell}_m}} = \frac{\varphi^{w(\tilde{\ell}_m)}_{\ell_k}}{\left(1 - \varphi^{w(\tilde{\ell}_m)}_{\tilde{\ell}_m}\right)}. \tag{B5}$$

Note that $\varsigma^{(\tilde{\ell}_m)}_{\ell_k}$ is not defined if $\varphi^{w(\tilde{\ell}_m)}_{\tilde{\ell}_m} = 1$. The pre-outage $\varphi^{w}_{\ell_k}$ and the post-outage $(\varphi^{w}_{\ell_k})^{(\tilde{\ell}_m)}$ relationship is given by [24]

$$\left(\varphi^{w}_{\ell_k}\right)^{(\tilde{\ell}_m)} \triangleq \varphi^{w}_{\ell_k} + \varsigma^{(\tilde{\ell}_m)}_{\ell_k}\varphi^{w}_{\tilde{\ell}_m} \quad \text{for any } w = \{i, j, t\}. \tag{B6}$$

We next generalize the *LODF* expression for the case of multiple line outages. We consider a set $\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)} = \{\tilde{\ell}_1, \ldots, \tilde{\ell}_\alpha\}$ of α outaged lines. We denote by $\underline{\boldsymbol{f}}_{(\alpha)} = [f_{\tilde{\ell}_1}, \ldots, f_{\tilde{\ell}_\alpha}]^T$ the vector of pre-outage real power flows on $\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}$ elements. For a line $\ell_k, \ell_k \notin \tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}$, we define the α-dimensional vector $\underline{\xi}^{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}}_{\ell_k}$, whose elements are the *LODF*s with the lines in $\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}$ outaged. We may view $\underline{\xi}^{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}}_{\ell_k}$ as providing the portions of the pre-outage flows $\underline{\boldsymbol{f}}_{(\alpha)}$ to flow on ℓ_k in the post-outage network

$$(\Delta f_{\ell_k})^{(\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)})} = \left[\underline{\xi}^{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}}_{\ell_k}\right]^T \underline{f}_{(\alpha)}. \tag{B7}$$

The derivation of the elements of $\underline{\xi}^{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}}_{\ell_k}$ is motivated by considering the case of the single line $\tilde{\ell}_1 = (\tilde{i}_1, \tilde{j}_1)$ outage. In the post-outage network, the line $\tilde{\ell}_1$ outage changes the real power flow on each line connected to node $\tilde{i}_1$ by the fraction of $f_{\tilde{\ell}_1}$ that is redistributed onto that line. Similarly, the line $\tilde{\ell}_1$ outage changes the real power flows on the lines connected to node $\tilde{j}_1$ by the respective fractions of $(-f_{\tilde{\ell}_1})$. Next, we consider the transaction $w(\tilde{\ell}_1) = \{\tilde{i}_1, \tilde{j}_1, t(\tilde{\ell}_1)\}$. The impact of the $t(\tilde{\ell}_1)$ is to add a flow $\varphi^{w(\tilde{\ell}_1)}_{\tilde{\ell}_1}t(\tilde{\ell}_1)$ on line $\tilde{\ell}_1$ and a net flow of $(1-\varphi^{w(\tilde{\ell}_1)}_{\tilde{\ell}_1})t(\tilde{\ell}_1)$ on all the other lines but $\tilde{\ell}_1$ connected to node $\tilde{i}_1$. We can, however, select the transaction amount $t(\tilde{\ell}_1)$ to be $(1 - \varphi^{w(\tilde{\ell}_1)}_{\tilde{\ell}_1})t(\tilde{\ell}_1) = f_{\tilde{\ell}_1}$. Since this impact on all the lines but $\tilde{\ell}_1$ is identical to that of the outage of $\tilde{\ell}_1$ on the same set of lines, we may simulate the outage impacts by introducing $w(\tilde{\ell}_1)$ with $t(\tilde{\ell}_1)$ given by $(1 - \varphi^{w(\tilde{\ell}_1)}_{\tilde{\ell}_1})t(\tilde{\ell}_1) = f_{\tilde{\ell}_1}$. This thought process leads to the definition of the *LODF* $\varsigma^{(\tilde{\ell}_1)}_{\ell_k}$.

The generalization of *LODF* to the multiple line outages in $\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}$ proceeds exactly along the same steps but explicitly takes into account the interactions between the impacts of the outaged lines. The simulation of the α outages is done by introducing the α transactions between the terminal nodes of each outaged line with the amounts determined so that

$$\underline{\boldsymbol{H}}_\alpha \underline{\boldsymbol{t}}_{(\alpha)} = \underline{\boldsymbol{f}}_{(\alpha)} \tag{B8}$$

where $\underline{\boldsymbol{t}}_{(\alpha)} = [t(\tilde{\ell}_1), \ldots, t(\tilde{\ell}_\alpha)]^T$ and

$$\underline{\boldsymbol{H}}_\alpha \triangleq \left[\underline{\boldsymbol{I}}^\alpha - \underline{\boldsymbol{\Phi}}_{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}}\right] = \begin{bmatrix} 1 - \varphi^{w(\tilde{\ell}_1)}_{\tilde{\ell}_1} \cdots & -\varphi^{w(\tilde{\ell}_\alpha)}_{\tilde{\ell}_1} \\ \vdots \quad \ddots & \vdots \\ -\varphi^{w(\tilde{\ell}_1)}_{\tilde{\ell}_\alpha} & \cdots 1 - \varphi^{w(\tilde{\ell}_\alpha)}_{\tilde{\ell}_\alpha} \end{bmatrix}. \tag{B9}$$

Due to superposition, the impacts of the α transactions on the real power flow in the line ℓ_k is given by

$$(\Delta f_{\ell_k})^{(\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)})} = \left[\varphi_{\ell_k}^{w(\tilde{\ell}_1)}, \ldots, \varphi_{\ell_k}^{w(\tilde{\ell}_\alpha)}\right] \underline{\boldsymbol{t}}_{(\alpha)} \tag{B10}$$

but from (B8), $\underline{\boldsymbol{f}}_{(\alpha)}$ and $\underline{\boldsymbol{t}}_{(\alpha)}$ are related. If $\underline{\boldsymbol{H}}_\alpha$ is nonsingular, then we may express (B10) as

$$(\Delta f_{\ell_k})^{(\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)})} = \left[\varphi_{\ell_k}^{w(\tilde{\ell}_1)}, \ldots, \varphi_{\ell_k}^{w(\tilde{\ell}_\alpha)}\right] [\underline{\boldsymbol{H}}^\alpha]^{-1} \underline{\boldsymbol{f}}_{(\alpha)}. \tag{B11}$$

It follows therefore that

$$\underline{\boldsymbol{H}}_\alpha^T \underline{\xi}_{\ell_k}^{(\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)})} = \left[\varphi_{\ell_k}^{w(\tilde{\ell}_1)}, \ldots, \varphi_{\ell_k}^{w(\tilde{\ell}_\alpha)}\right]^T. \tag{B12}$$

Note that $\underline{\xi}_{\ell_k}^{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}}$ is defined if $\underline{\boldsymbol{H}}_\alpha$ is nonsingular and undefined otherwise. The expression (B12) appears to be new.

Since $\underline{\boldsymbol{\Phi}}_{\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}}$ is a submatrix of $\underline{\boldsymbol{\Phi}}_{\boldsymbol{\mathcal{L}}}$, its components satisfy (B4). Consequently, the components of $\underline{\boldsymbol{H}}_\alpha$ satisfy

$$1 > h_{i,i} \geqslant 0, h_{i,j}h_{j,i} \geqslant 0, \quad i,j = 1,\ldots,\alpha. \tag{B13}$$

APPENDIX C
PROOFS OF MATHEMATICAL STATEMENTS

Proof of Theorem 1

We use an inductive approach to prove part (*i*). We first illustrate that the statement is true for a minimal cutset, which has a single element. Consider the network in which $\beta - 1$ elements of $\hat{\boldsymbol{\mathcal{L}}}_{(\beta)}$ are outaged. There are β possible outage permutations, which we denote by $\hat{\boldsymbol{\mathcal{L}}}_{(\beta-1)}^j = \hat{\boldsymbol{\mathcal{L}}}_{(\beta)} \backslash \{\hat{\ell}_j\}, j = 1, \ldots, \beta$. In each outaged network, $\{\hat{\ell}_j\} \notin \hat{\boldsymbol{\mathcal{L}}}_{(\beta-1)}^j$ constitutes a minimal cutset, and from (6)

$$\left(\varphi_{\hat{\ell}_j}^{w(\hat{\ell}_k)}\right)^{(\hat{\boldsymbol{\mathcal{L}}}_{\beta-1}^j)} = 1 \quad \hat{\ell}_j \in \hat{\boldsymbol{\mathcal{L}}}_{(\beta)}, \quad \hat{\ell}_k \in \hat{\boldsymbol{\mathcal{L}}}_{(\beta)} \backslash \{\hat{\ell}_j\}. \tag{C1}$$

Next, we assume that the statement is true for the minimal cutset constituted by a set of $\beta - 1$ elements and prove its veracity for a set of β elements. Assume $(\varphi_{\hat{\ell}_j}^{w(\hat{\ell}_k)})^{(\hat{\ell}_k)} > 0$ holds $j \neq k, \hat{\ell}_j, \hat{\ell}_k \in \hat{\boldsymbol{\mathcal{L}}}_{(\beta)}$. By (B6), we can state that

$$\begin{aligned}\left(\varphi_{\tilde{\ell}_j}^{w(\hat{\ell}_k)}\right)^{(\hat{\ell}_k)} &= \varphi_{\tilde{\ell}_j}^{w(\hat{\ell}_k)} + \varsigma_{\tilde{\ell}_j}^{(\hat{\ell}_k)} \varphi_{\tilde{\ell}_k}^{w(\hat{\ell}_k)} \\ &= \varphi_{\tilde{\ell}_j}^{w(\hat{\ell}_k)} \left(1 + \frac{\varphi_{\tilde{\ell}_k}^{w(\hat{\ell}_k)}}{1 - \varphi_{\tilde{\ell}_k}^{w(\hat{\ell}_k)}}\right), \cdot \hat{\ell}_j, \hat{\ell}_k \in \hat{\boldsymbol{\mathcal{L}}}_{(\beta)}, j \neq k.\end{aligned} \tag{C2}$$

It follows form that (B4) $\varphi_{\tilde{\ell}_k}^{w(\tilde{\ell}_k)} > 0$; hence, (C2) implies that

$$\varphi_{\tilde{\ell}_j}^{w(\tilde{\ell}_k)} > 0, \hat{\ell}_j, \hat{\ell}_k \in \hat{\boldsymbol{\mathcal{L}}}_{(\beta)}, j \neq k. \tag{C3}$$

Since any transaction between the subnetworks $\boldsymbol{\mathcal{G}_a}$ and $\boldsymbol{\mathcal{G}_b}$ must flow over the minimal cutset elements, the proof of (*ii*) follows from the results in part (*i*) and (8).

For the proof of (iii), let $w^a = \{\hat{i}_m, \hat{i}_k, t\}$ be an intra-subnetwork transaction in $\boldsymbol{\mathcal{N}_a}$. For an inter-subnetwork transaction $w' = \{\hat{i}_m, \hat{j}_k, t\}, \hat{j}_k \in \boldsymbol{\mathcal{N}}_b$, from part (*ii*)

$$\sum_{\hat{\ell}_k \in \hat{\boldsymbol{\mathcal{L}}}} \varphi_{\hat{\ell}_k}^{w'} = 1. \tag{C4}$$

Similarly, for the inter-subnetwork transaction $w'' = \{\hat{j}_k, \hat{i}_k, t\}$

$$\sum_{\hat{\ell}_k \in \hat{\boldsymbol{\mathcal{L}}}} \varphi_{\hat{\ell}_k}^{w''} = -1. \tag{C5}$$

Note that w^a is equivalent to the two transactions w' and w'' and due to linearity

$$\sum_{\hat{\ell}_k \in \hat{\boldsymbol{\mathcal{L}}}} \varphi_{\hat{\ell}_k}^{w^a} = \sum_{\hat{\ell}_k \in \hat{\boldsymbol{\mathcal{L}}}} \varphi_{\hat{\ell}_k}^{w'} + \sum_{\hat{\ell}_k \in \hat{\boldsymbol{\mathcal{L}}}} \varphi_{\hat{\ell}_k}^{w''} = 0. \tag{C6}$$

A similar argument holds for an arbitrary intra-subnetwork where terminal node pairs are in $\boldsymbol{\mathcal{N}_b}$.

Proof of Theorem 2

We prove the necessary condition by contradiction. Let $\underline{\boldsymbol{H}}_\alpha$ be singular and assume $\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}$ does not contain a minimal cutset. We perform the Gaussian elimination on $\underline{\boldsymbol{H}}_\alpha$ and compute, at step $k, k = 1, \ldots, \alpha, i = k, \ldots, \alpha$

$$h_{i,i}^{(k)} = h_{i,i}^{(k-1)} - \frac{h_{i,k-1}^{(k-1)}}{h_{k-1,k-1}^{(k-1)}} h_{k-1,i}^{(k-1)} = 1 - \left(\varphi_{\tilde{\ell}_i}^{w(\tilde{\ell}_i)}\right)^{(\tilde{\boldsymbol{\mathcal{L}}}_{(k-1)})} \tag{C7}$$

where we use the relation given in (B6). The Gaussian elimination of the singular matrix $\underline{\boldsymbol{H}}_\alpha$ results in a zero pivot at some elimination step $m \leqslant \alpha$ [23], which can only happen if $(\varphi_{\tilde{\ell}_m}^{w(\tilde{\ell}_m)})^{(\tilde{\boldsymbol{\mathcal{L}}}_{(m-1)})} = 1$. By (5), $\{\tilde{\ell}_m\}$ constitutes a minimal cutset of the network in which first $m-1$ lines of $\tilde{\boldsymbol{\mathcal{L}}}_{(\alpha)}$ are outaged. Thus, $\{\tilde{\ell}_1, \tilde{\ell}_2, \ldots, \tilde{\ell}_m\} \subseteq \tilde{\boldsymbol{\mathcal{L}}}$ contains a minimal cutset that is a contradiction of the original problem.

For the sufficiency condition, let $\hat{\boldsymbol{\mathcal{L}}} \subseteq \tilde{\boldsymbol{\mathcal{L}}}$ be a minimal cutset with $\hat{\boldsymbol{\mathcal{L}}} = \tilde{\boldsymbol{\mathcal{L}}}_{(m)} = \{\tilde{\ell}_1, \ldots, \tilde{\ell}_m\}, 1 \leqslant m \leqslant \alpha$. We partition $\underline{\boldsymbol{H}}_\alpha$

$$\underline{\boldsymbol{H}}_\alpha = \begin{bmatrix} \underline{\boldsymbol{H}}_m & \underline{\boldsymbol{C}} \\ \underline{\boldsymbol{B}} & \underline{\boldsymbol{D}} \end{bmatrix}. \tag{C8}$$

Since $\underline{\boldsymbol{H}}_\alpha$ satisfies (B13), so does $\underline{\boldsymbol{H}}_m$. We construct the diagonal $\underline{\boldsymbol{U}} = \mathrm{diag}\{u_1, \ldots, u_m\}$ with $u_iu_j = \mathrm{sign}\{-h_{i,j}\}, |u_i| = 1$ and $i \neq j$. Thus, $u_iu_jh_{i,j} = -|h_{i,j}| = -|\varphi_{\tilde{\ell}_i}^{w(\tilde{\ell}_j)}| < 0, i,j = 1, \ldots, m, i \neq j$. The diagonal elements of $\underline{\boldsymbol{H}}_m^\dagger = \underline{\boldsymbol{U}}\underline{\boldsymbol{H}}_m\underline{\boldsymbol{U}}$ are given by $u_i^2 h_{i,i} = 1 - \varphi_{\tilde{\ell}_i}^{w(\tilde{\ell}_i)} > 0$. Algebraic sum of column j

$$\begin{aligned}\sum_{i=1}^m u_iu_jh_{i,j} &= -\sum_{i=1, i\neq j}^m \left|\varphi_{\tilde{\ell}_i}^{w(\tilde{\ell}_j)}\right| + 1 - \varphi_{\tilde{\ell}_j}^{w(\tilde{\ell}_j)} \\ &= -\sum_{i=1}^m \left|\varphi_{\tilde{\ell}_i}^{w(\tilde{\ell}_j)}\right| + 1 = 0\end{aligned} \tag{C9}$$

and so $(\underline{\boldsymbol{1}}^m)^T \underline{\boldsymbol{H}}_m^\dagger = (\underline{\boldsymbol{0}})^T$. Also the components of $\underline{\boldsymbol{UC}}$ are given by $u_ih_{i,k}, k = m+1, \ldots, \alpha, i = 1, \ldots, m$. Theorem 1 implies that $\sum_{i=1}^m u_ih_{i,k} = 0$, and so, $(\underline{\boldsymbol{1}}^m)^T \underline{\boldsymbol{UC}} = (\underline{\boldsymbol{0}}^m)^T$.

Consider $\underline{\boldsymbol{H}}^{\dagger}_{\alpha} = \tilde{\underline{\boldsymbol{U}}}\underline{\boldsymbol{H}}_{\alpha}\tilde{\underline{\boldsymbol{U}}}$ with $\tilde{\underline{\boldsymbol{U}}} = \mathrm{diag}\{\underline{\boldsymbol{U}}, \underline{\boldsymbol{I}}^{\alpha-k}\}$. Then $\tilde{\underline{\boldsymbol{v}}}^T \underline{\boldsymbol{H}}^{\dagger}_{\alpha} = (\underline{\mathbf{0}}^{\alpha})^T$ for the vector $\tilde{\underline{\boldsymbol{v}}} = [(\underline{\mathbf{1}}^m)^T(\underline{\mathbf{0}}^{\alpha-m})^T]^T$.

In other words, we construct a vector $\tilde{\underline{\boldsymbol{v}}} \neq \underline{\mathbf{0}}$ and $\tilde{\underline{\boldsymbol{v}}} \in \boldsymbol{\mathcal{R}}(\underline{\boldsymbol{H}}^T_{\alpha})$. The rows of $\underline{\boldsymbol{H}}^{\dagger}_{\alpha}$ form a linearly dependent set indicating that $\underline{\boldsymbol{H}}^{\dagger}_{\alpha}$ is singular. Therefore, $\underline{\boldsymbol{H}}_{\alpha} = \tilde{\underline{\boldsymbol{U}}}^{-1}\underline{\boldsymbol{H}}^{\dagger}_{\alpha}\tilde{\underline{\boldsymbol{U}}}^{-1}$ is also singular. Furthermore, $\underline{\boldsymbol{v}} = \tilde{\underline{\boldsymbol{U}}}^{-1}\tilde{\underline{\boldsymbol{v}}} \neq \underline{\mathbf{0}}^{\alpha}$ and $\underline{\boldsymbol{v}}^T\underline{\boldsymbol{H}}_{\alpha} = (\underline{\mathbf{0}}^{\alpha})^T$. Since $|v_i| = |(u_i)^{-1}\tilde{v}_i| = |\tilde{v}_i|$, then using the statement of $\underline{\boldsymbol{H}}_{\alpha}$

$$|v_i| = \begin{cases} 1, & \forall \tilde{\ell}_i \in \hat{\boldsymbol{\mathcal{L}}} \\ 0, & \forall \tilde{\ell}_i \notin \hat{\boldsymbol{\mathcal{L}}}. \end{cases} \tag{C10}$$

RRQR Factorization

We consider $\underline{\boldsymbol{H}}_{\alpha}$ with such $\boldsymbol{\mathcal{R}}(\underline{\boldsymbol{H}}^T_{\alpha}) = \rho(\alpha)$. The set of columns of $\underline{\boldsymbol{Q}}_{\rho(\alpha)}$ and $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_{\rho(\alpha)}\}$ form a basis for $\boldsymbol{\mathcal{R}}(\underline{\boldsymbol{H}}^T_{\alpha})$. Therefore, $\underline{\boldsymbol{Q}}_{\rho(\alpha)}$ and $[\underline{\boldsymbol{v}}_1 \vdots \underline{\boldsymbol{v}}_2 \vdots \cdots \vdots \underline{\boldsymbol{v}}_{\rho(\alpha)}]$ are full-rank arrays. We define $\underline{\boldsymbol{T}}$ to be the matrix in $\mathbb{R}^{\rho(\alpha)\times\rho(\alpha)}$ relating the two set of vectors: $[\underline{\boldsymbol{v}}_1 \vdots \underline{\boldsymbol{v}}_2 \vdots \cdots \vdots \underline{\boldsymbol{v}}_{\rho(\alpha)}]\underline{\boldsymbol{T}} = \underline{\boldsymbol{Q}}_{\rho(\alpha)}$. Since the components of each $\underline{\boldsymbol{v}}_i i = 1, \ldots, \rho(\alpha)$ satisfy (C10), we can reorder the rows of $[\underline{\boldsymbol{v}}_1 \vdots \underline{\boldsymbol{v}}_2 \vdots \cdots \vdots \underline{\boldsymbol{v}}_{\rho(\alpha)}]$, introduce multiplication by -1 where necessary, and perform identical operations on the corresponding rows of $\underline{\boldsymbol{Q}}_{\rho(\alpha)}$ to obtain the first $\rho(\alpha)$ elements as the identity matrix. We denote the other rows of $[\underline{\boldsymbol{v}}_1 \vdots \underline{\boldsymbol{v}}_2 \vdots \cdots \vdots \underline{\boldsymbol{v}}_{\rho(\alpha)}]$ by $\underline{\boldsymbol{V}}$. Then the transformation that relates the rearranged rows of the two rearranged bases is

$$\begin{bmatrix} \underline{\boldsymbol{Q}}'_{\rho(\alpha)} \\ \underline{\boldsymbol{Q}}''_{\rho(\alpha)} \end{bmatrix} = \begin{bmatrix} \underline{\boldsymbol{I}}^{\rho(\alpha)} \\ \underline{\boldsymbol{V}} \end{bmatrix} \underline{\boldsymbol{T}} = \begin{bmatrix} \hat{\underline{\boldsymbol{v}}}_1 \vdots \hat{\underline{\boldsymbol{v}}}_2 \vdots \cdots \vdots \hat{\underline{\boldsymbol{v}}}_{\rho(\alpha)} \end{bmatrix} \tag{C11}$$

where each $\hat{\underline{\boldsymbol{v}}}_i$ also satisfies (C10). So, $\underline{\boldsymbol{T}} = \underline{\boldsymbol{Q}}'_{\rho(\alpha)}$.

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