Slides taken from:

EEE 598
Electric Energy Markets

School of ECEE, ASU

Professor Kory W. Hedman

Duality and
the Dual of the DCOPF

Important Note

- There are many different ways to formulate a DCOPF Problem
- Today, we will examine one particular formulation and its corresponding dual
- We will discuss the economic interpretation of this dual problem, its variables, and its constraints
- However, it is important to understand that the interpretations we develop apply to this DCOPF formulation
- If you take a dual of a different DCOPF formulation, the dual will not be the same and you may get different interpretations of that different dual

DCOPF Formulation

DCOPF Formulation

- **Nomenclature**
- ***Indices and Sets***
- g: Generator.
- $g(n)$: Set of generators at bus n.
- k: Transmission element (line or transformer).
- $k(n,.)$: Set of transmission assets with n as the ‘to’ node.
- $k(.,n)$: Set of transmission assets with n as the ‘from’ node.
- m, n: Buses (nodes).
- ***Parameters***
- B_k: Susceptance of transmission element k.
- c_g: Production cost for generator g.
- d_n: Real power load at node n.
- $P^{max}{}_g$, $P^{min}{}_g$: Max and min capacity of generator g.
- $P^{max}{}_k$, $P^{min}{}_k$: Max and min capacity rating of transmission element k. Generally, $P^{min}{}_k = -P^{max}{}_k$.
- ***Variables***
- P_g: Real power supply from generator g at node n.
- P_k: Real power flow from node m to node n for transmission element k.
- θ_n: Bus voltage angle at node n.

DCOPF Formulation

- Objective:
- Minimize Total Cost: $$\sum_g c_g P_g$$
- For this DCOPF formulation, we will assume that generator costs are linear

DCOPF Formulation

- Generator Lower and Upper Bounds:

$$-P_g \geq -P_g^{\max}, \forall g$$

$$P_g \geq 0, \forall g$$

- For simplicity of the derivation that we will cover today in class, we will assume that the lower bound of the generator production is **ZERO**

DCOPF Formulation

- Power Flow Limits (thermal or stability) on Line *k*:

$$-P_k \geq -P_k^{\max}, \forall k$$ Upper Bound

$$P_k \geq -P_k^{\max}, \forall k$$ Lower Bound

- For simplicity, we will assume:
  - $P_k^{min} = -P_k^{max}$

DCOPF Formulation

- Power Flow Equation for Line *k:*

$$B_k(\theta_n - \theta_m) - P_k = 0, \forall k$$

- This is a linearized approximation to the actual AC OPF line flow constraint
- Specifies that the flow on line *k* is equal to the Susceptance of line *k* times the [*'to bus'* voltage angle variable – *'from bus'* voltage angle variable]

DCOPF Formulation

- Node Balance Constraint:

$$\sum_{\forall k(n,.)} P_k - \sum_{\forall k(.,n)} P_k + \sum_{\forall g(n)} P_g = d_n, \forall n$$

- Power flow into bus *n* must equal the power flow out of bus *n*
- Injections are positive, withdrawals are negative
- First term: sum $_{\text{over } k(n,*)}$ P_k
  - For all lines that have *'to bus'* = *n*
- Second term: sum $_{\text{over } k(*,n)}$ P_k
  - For all lines that have *'from bus'* = *n*
- Third term: sum $_{\text{over } g(n)}$ P_g
  - For multiple generators at one bus, calculate the total production at that location
- Right Hand Side: Demand at bus *n*

Full DCOPF Formulation

Primal Problem:

Minimize: $\sum_{g} c_g P_g$

$s.t.$

$$\sum_{\forall k(n,.)} P_k - \sum_{\forall k(.,n)} P_k + \sum_{\forall g(n)} P_g = d_n, \forall n \qquad (LMP_n)$$

$$-P_k \geq -P_k^{\max}, \forall k \qquad (F^+_k)$$

$$P_k \geq -P_k^{\max}, \forall k \qquad (F^-_k)$$

$$B_k(\theta_n - \theta_m) - P_k = 0, \forall k \qquad (S_k)$$

$$-P_g \geq -P_g^{\max}, \forall g \qquad (\alpha_g)$$

$P_g \geq 0$; θ_n, P_k free

Break

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Dual of the DCOPF: Objective

Dual of this particular DCOPF

DCOPF Formulation

- Dual Variables:
- ***LMP_n*:** Dual variable on bus *n*'s power balance constraint (**locational marginal price**).
- ***F^+_k, F^-_k*:** Dual variables on transmission element *k*'s capacity constraints, upper bound and lower bound respectively (**flowgate marginal prices**).
- ***S_k*:** Dual variable on transmission line *k*'s power flow constraint (**susceptance price**).
- **α_g:** Dual variable on generator *g*'s capacity constraint, upper bound constraint.

Primal Problem

Primal Problem:

Minimize: $\sum_{g} c_g P_g$

$s.t.$

$$\sum_{\forall k(n,.)} P_k - \sum_{\forall k(.,n)} P_k + \sum_{\forall g(n)} P_g = d_n, \forall n \qquad (LMP_n)$$

$$-P_k \geq -P_k^{\max}, \forall k \qquad (F^{+}_k)$$

$$P_k \geq -P_k^{\max}, \forall k \qquad (F^{-}_k)$$

$$B_k(\theta_n - \theta_m) - P_k = 0, \forall k \qquad (S_k)$$

$$-P_g \geq -P_g^{\max}, \forall g \qquad (\alpha_g)$$

$P_g \geq 0$; θ_n, P_k free

Objective of the Dual

- First Primal Constraint:

$$\sum_{\forall k(n,.)} P_k - \sum_{\forall k(.,n)} P_k + \sum_{\forall g(n)} P_g = d_n, \forall n \qquad (LMP_n)$$

  - $LMP_n * d_n$

- First Dual Objective Term: $\sum_n d_n LMP_n$

Objective of the Dual

- Second Primal Constraint:

$$-P_k \geq -P_k^{\max}, \forall k \qquad (F^+_k)$$

  - $-P_k^{max} * F^+_k$

- Second Dual Objective Term:

$$-\sum_k P_k^{\max}\left(F_k^+\right)$$

Objective of the Dual

- Third Primal Constraint:

$$P_k \geq -P_k^{\max} \ , \forall k \qquad (F^-_k)$$

  - $-P_k^{max} * F^-_k$

- Third Dual Objective Term:

$$-\sum_k P_k^{\max}\left(F_k^-\right)$$

Objective of the Dual

- Fourth Primal Constraint:

$$B_k(\theta_n - \theta_m) - P_k = 0, \forall k \qquad (S_k)$$

  - $0 * S_k$

- Dual Objective Term:
  - 0

Objective of the Dual

- Fifth Primal Constraint:

$$-P_g \geq -P_g^{\max}, \forall g \qquad (\alpha_g)$$

  - $-P^{max}{}_g \alpha_g$

- Dual Objective Term: $$-\sum_g P_g^{\max} \alpha_g$$

Objective of the Dual

- Total Objective of the Dual:

$$\sum_n d_n LMP_n - \sum_k P_k^{\max}\left(F_k^+ + F_k^-\right) - \sum_g P_g^{\max}\alpha_g$$

Break

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Dual of the DCOPF: Constraints

Dual of this particular DCOPF

Dual Constraint Corresponding to the Generator Output Variable

- Primal constraints involving P_g
  - Left hand side of the dual constraint will have LMP_n

$$\sum_{\forall k(n,.)} P_k - \sum_{\forall k(.,n)} P_k + \sum_{\forall g(n)} P_g = d_n, \forall n \qquad (LMP_n)$$

  - Left hand side of the dual constraint will have $-\alpha_g$

$$-P_g \geq -P_g^{\max}, \forall g \qquad (\alpha_g)$$

- Primal Objective:
  - Right hand side of the dual constraint for P_g will have c_g

$$\sum_g c_g P_g$$

Final Dual Constraint Corresponding to the Generator Output Variable

- Sign restriction on P_g: P_g >= 0
  - Constraint form is **<=** in Dual (a max)

$$LMP_n - \alpha_g \leq c_g, \forall g \qquad (P_g)$$

- $\mathrm{LMP}_{n(g)}$

Dual Constraint Corresponding to the Power Line Flow Variable

- Primal constraints involving P_k

$$\sum_{\forall k(n,.)} P_k - \sum_{\forall k(.,n)} P_k + \sum_{\forall g(n)} P_g = d_n, \forall n \qquad (LMP_n)$$

- Suppose P_k is connected to bus 5 (*n – 'to bus'*) from bus 4 (*m – 'from bus'*)
  - For the node balance constraint for bus 5, on the left hand side will be: P_k
  - For the node balance constraint for bus 4, on the left hand side will be: $-P_k$
- The left hand side of the dual constraint for P_k will include:
  - $\boldsymbol{LMP_n - LMP_m}$

Dual Constraint Corresponding to the Power Line Flow Variable

- Primal constraints involving P_k
- Thermal/Stability Capacity Limit (Upper):

$$-P_k \geq -P_k^{\max}, \forall k \qquad (F^+_k)$$

  - Left hand side of the dual constraint for P_k will include: $-F^+_k$
- Thermal/Stability Capacity Limit (Lower Bound):

$$P_k \geq -P_k^{\max}, \forall k \qquad (F^-_k)$$

  - Left hand side of the dual constraint for P_k will include: $+F^-_k$

Dual Constraint Corresponding to the Power Line Flow Variable

- Primal constraints involving P_k
- Line flow equation for P_k:

$$B_k(\theta_n - \theta_m) - P_k = 0, \forall k \qquad (S_k)$$

  – Left hand side of the dual constraint for P_k will include: $\boldsymbol{-S_k}$

Final Dual Constraint Corresponding to the Power Line Flow Variable

- Primal objective does not include P_k
  - Right hand side of dual constraint is **ZERO**
- Sign restriction on P_k: P_k Free
  - Constraint form is **=** in Dual (a max)

$$LMP_n - LMP_m - F_k^{+} + F_k^{-} - S_k = 0, \forall k \quad \left(P_k\right)$$

Dual Constraint Corresponding to the Bus Voltage Angle Variable

- Primal constraints involving θ_n

$$B_k(\theta_n - \theta_m) - P_k = 0, \forall k \qquad (S_k)$$

- Suppose 4 lines are connected to bus *n*: Lines *A*, *B*, *C*, *D*
  - Lines *A* and *D* have bus *n* defined as a *'to bus'*
  - Lines *B* and *C* have bus *n* defined as a *'from bus'*
- For lines *A* and *D*, in the above equation we have: $B_k \theta_n$
- For lines *B* and *C*, in the above equation we have: $-B_k \theta_n$
- So the left hand side of the dual will contain:
  - For lines *A* and *D*: $\boldsymbol{B_k S_k}$
  - For lines *B* and *C*: $\boldsymbol{-B_k S_k}$

Dual Constraint Corresponding to the Bus Voltage Angle Variable

- Primal constraints involving θ_n

$$B_k(\theta_n - \theta_m) - P_k = 0, \forall k \qquad (S_k)$$

- Continue from last slide
- So if *n* is the *'to bus'* for line *k*, we have $\mathbf{B_k S_k}$
  - Sum (for all lines with *n* = *'to bus'*) all of these terms to get: $$\sum_{k(n,.)} B_k S_k$$
- If *n* is the *'from bus'* for line *k*, we have $\mathbf{-B_k S_k}$
  - Sum (for all lines with *n* = *'from bus'*) all of these terms to get: $$-\sum_{k(.,n)} B_k S_k$$

Final Dual Constraint Corresponding to the Bus Voltage Angle Variable

- Primal objective does not include θ_n
  - Right hand side of dual constraint is **ZERO**
- Sign restriction on θ_n: θ_n Free
  - Constraint form is **=** in Dual (a max)

$$\sum_{k(n,.)} B_k S_k - \sum_{k(.,n)} B_k S_k = 0, \forall n \qquad (\theta_n)$$

Break

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Dual of the DCOPF: Sign Restrictions and Complete Formulation

Dual of this particular DCOPF

Sign Restrictions on the Dual Variables

- LMP_n: Free
- $F^+{}_k$ >= 0
- $F^-{}_k$ >= 0
- S_k: Free
- α_g >= 0

Primal Problem:

Minimize: $\sum_g c_g P_g$

$$
\begin{aligned}
&s.t. \\
&\sum_{\forall k(n,.)} P_k - \sum_{\forall k(.,n)} P_k + \sum_{\forall g(n)} P_g = d_n, \forall n && (LMP_n) \\
&-P_k \geq -P_k^{\max}, \forall k && (F^+{}_k) \\
&P_k \geq -P_k^{\max}, \forall k && (F^-{}_k) \\
&B_k(\theta_n - \theta_m) - P_k = 0, \forall k && (S_k) \\
&-P_g \geq -P_g^{\max}, \forall g && (\alpha_g) \\
&P_g \geq 0;\ \theta_n,\ P_k \text{ free}
\end{aligned}
$$

Complete Dual Formulation

Dual Problem:

Maximize: $\sum_{n} d_n LMP_n - \sum_{k} P_k^{\max}\left(F_k^+ + F_k^-\right) - \sum_{g} P_g^{\max}\alpha_g$

s.t.

$$LMP_n - LMP_m - F_k^+ + F_k^- - S_k = 0, \forall k \qquad (P_k)$$

$$LMP_n - \alpha_g \le c_g, \forall g \qquad (P_g)$$

$$\sum_{k(n,.)} B_k S_k - \sum_{k(.,n)} B_k S_k = 0, \forall n \qquad (\theta_n)$$

$F^+{}_k$, $F^-{}_k$, $\alpha_g \ge 0$; LMP_n, S_k free

Review of Primal Problem

Primal Problem:

Minimize: $\sum_{g} c_g P_g$

$s.t.$

$$\sum_{\forall k(n,.)} P_k - \sum_{\forall k(.,n)} P_k + \sum_{\forall g(n)} P_g = d_n, \forall n \qquad (LMP_n)$$

$$-P_k \geq -P_k^{\max}, \forall k \qquad (F^+_k)$$

$$P_k \geq -P_k^{\max}, \forall k \qquad (F^-_k)$$

$$B_k(\theta_n - \theta_m) - P_k = 0, \forall k \qquad (S_k)$$

$$-P_g \geq -P_g^{\max}, \forall g \qquad (\alpha_g)$$

$P_g \geq 0$; θ_n, P_k free

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Dual of the DCOPF: Analyzing Dual Variables and Constraints Part 1

Dual of this particular DCOPF

Complete Dual Formulation

Dual Problem:

Maximize: $\sum_{n} d_n LMP_n - \sum_{k} P_k^{\max}\left(F_k^+ + F_k^-\right) - \sum_{g} P_g^{\max}\alpha_g$

s.t.

$$LMP_n - LMP_m - F_k^+ + F_k^- - S_k = 0, \forall k \qquad (P_k)$$

$$LMP_n - \alpha_g \le c_g, \forall g \qquad (P_g)$$

$$\sum_{k(n,.)} B_k S_k - \sum_{k(.,n)} B_k S_k = 0, \forall n \qquad (\theta_n)$$

$F^+{}_k$, $F^-{}_k$, $\alpha_g \ge 0$; LMP_n, S_k free

Locational Marginal Price

- Dual variable of the Node Balance Constraint
- Specifies the increase (decrease) to the primal objective if there is slightly more (less) consumption at bus *n*

$$\sum_{\forall k(n,.)} P_k - \sum_{\forall k(.,n)} P_k + \sum_{\forall g(n)} P_g = d_n, \forall n \qquad (LMP_n)$$

Dual Variable of the Generator Upper Bound Constraint

- Interpretation: $-P_g \geq -P_g^{\max}, \forall g \qquad (\alpha_g)$
- α_g: non-negative (for this dual)
- α_g: Reflects the marginal value of increasing the capacity of generator *g*
- α_g : Decreasing the right hand side of this equation decreases (or doesn't change) the primal objective
- Complementary slackness (CS) tells us that $\alpha_g = 0$ if $P_g < P^{max}_g$ (or $-P_g > -P^{max}_g$)

Dual Constraint Corresponding to the Generator Output Variable

$$LMP_n - \alpha_g \leq c_g, \forall g \qquad \left(P_g\right)$$

- Suppose $P_g < P^{max}_g$ (then $\alpha_g = 0$)
  - Suppose $P_g > 0$, by CS, $LMP_n = c_g$
  - Interpretation 1: If a generator (that has a lower bound of zero) is operating and it is not operating at its capacity, then the LMP at its bus is equal to its marginal cost
  - What would change to this interpretation if this generator had a lower bound that was not zero?

Dual Constraint Corresponding to the Generator Output Variable

$$LMP_n - \alpha_g \le c_g, \forall g \qquad \left(P_g\right)$$

- Suppose $P_g < P^{max}_g$ (then $\alpha_g = 0$)
  - Suppose $P_g = 0$, then $LMP_n <= c_g$
  - Interpretation 2: If a generator has a lower bound of zero and it is not producing anything, then the LMP at its bus is less than or equal to its marginal cost
- How does unit commitment affect this interpretation? What about our assumption that the lower bound is ZERO?
  - You may apply this interpretation incorrectly if you forget about the assumptions that we have made and/or you apply this to a unit commitment problem

Dual Constraint Corresponding to the Generator Output Variable

$$LMP_n - \alpha_g \le c_g, \forall g \qquad \left(P_g\right)$$

- Suppose $P_g = P^{max}_g$ (then $\alpha_g >= 0$)
  - First, CS tells us that $LMP_n - \alpha_g = c_g$
  - Interpretation 3: If a generator is operating at its upper bound, then the LMP at its bus may equal its marginal cost (if $\alpha_g = 0$) or the LMP may exceed its marginal cost (if $\alpha_g > 0$)
  - The dual variable α_g reflects the value of increasing the generator's capacity and this value is the difference between the local LMP and the generator's cost. Why is this the case?
    - The value of adding capacity to this generator cannot exceed the difference between the LMP and the marginal cost because this bus can always buy another MW from the grid for the LMP price so this places a cap on the value of an increase in the generator's capacity

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Dual of the DCOPF: Analyzing Dual Variables and Constraints Part 2

Dual of this particular DCOPF

Complete Dual Formulation

Dual Problem:

Maximize: $\sum_n d_n LMP_n - \sum_k P_k^{\max}\left(F_k^+ + F_k^-\right) - \sum_g P_g^{\max}\alpha_g$

s.t.

$$LMP_n - LMP_m - F_k^+ + F_k^- - S_k = 0, \forall k \qquad (P_k)$$

$$LMP_n - \alpha_g \le c_g, \forall g \qquad (P_g)$$

$$\sum_{k(n,.)} B_k S_k - \sum_{k(.,n)} B_k S_k = 0, \forall n \qquad (\theta_n)$$

$F^+{}_k$, $F^-{}_k$, $\alpha_g \geq 0$; LMP_n, S_k free

Dual Variables on the Line Flow Capacity Constraints

$$-P_k \ge -P_k^{\max}, \forall k \quad (F^+_k) \qquad P_k \ge -P_k^{\max}, \forall k \quad (F^-_k)$$

- ***F^+_k, F^-_k*: Flowgate Marginal Prices**
- F^+_k, F^-_k are both non-negative (for this dual)
- Decrease the right hand side of both equations will decrease (or not change) the primal objective
- CS tells us that $F^+_k * F^-_k = 0$
- CS tells us that $F^+_k = 0$ if $P_k < P_k^{max}$
- CS tells us that $F^-_k = 0$ if $P_k > -P_k^{max}$

Dual Variables on the Line Flow Equation

$$B_k(\theta_n - \theta_m) - P_k = 0, \forall k \qquad (S_k)$$

- **S_k: Susceptance price**
  - May also be referred to as an Admittance price
- Since the primal equation is an equality, CS tells us nothing about the sign of S_k
- S_k can be negative, zero, or positive

Dual Constraint Corresponding to the Power Line Flow Variable

$$LMP_n - LMP_m - F_k^{+} + F_k^{-} - S_k = 0, \forall k \quad \left(P_k\right)$$

- Interpretation 1: Assume line is not congested: $F^{+}_k = 0 = F^{-}_k$
- There can be a difference in LMP for two buses (*n* and *m*) that are connected by line *k* even if the line is not congested ($F^{+}_k = 0 = F^{-}_k$)
  - Because S_k can be non-zero
- Economic interpretation of S_k:
  - S_k is non-zero when the flow on line *k* is restricted due to the impedance of line *k* in relation to another line in the network
  - Another line is congested, which limits the flow on line *k*
  - We may be able to send more power across line *k* if the impedance was changed
  - This is what causes a non-zero value for S_k and this is why it is called an ***admittance price or susceptance price***

Dual Constraint Corresponding to the Power Line Flow Variable

$$LMP_n - LMP_m - F_k^+ + F_k^- - S_k = 0, \forall k \quad \left(P_k\right)$$

- Interpretation 2: Assume $S_k = 0$
- If $S_k = 0$ and $LMP_n > LMP_m$
  - Then: $F^-_k = 0$ and $F^+_k = LMP_n - LMP_m$
- If $S_k = 0$ and $LMP_n < LMP_m$
  - Then: $F^+_k = 0$ and $F^-_k = LMP_m - LMP_n$
- If $S_k = 0$, then the Flowgate Marginal Price of line k = (absolute) LMP difference across the line
- This also states that if $S_k = 0$, then the flow of energy must be in the direction of the increase in LMP
  - There can only be a flow from an expensive bus to a cheap bus if S_k is non-zero
- **Note: these are results for a DC, lossless model**

Dual Constraint Corresponding to the Power Line Flow Variable

$$LMP_n - LMP_m - F_k^{+} + F_k^{-} - S_k = 0, \forall k \quad \left(P_k\right)$$

- Interpretation 3: Assume LMP_n - LMP_m = 0
  - It is still possible for the line to be congested and for there to be a susceptance price
- Interpretation 4: Without any assumption:
  - The difference in LMP across a line is related to the Flowgate marginal price and the Susceptance price
- Note that this is based on this particular formulation; you can rewrite many of these equations differently (multiply the line flow constraint by -1)

Dual Constraint Corresponding to the Bus Voltage Angle Variable

$$\sum_{k(n,.)} B_k S_k - \sum_{k(.,n)} B_k S_k = 0, \forall n \quad (\theta_n)$$

- Interpretation 1: For any bus, Sum of[Susceptance prices for the *'to bus'* lines connected to bus *n* times the Susceptance value for that line] must equal the Sum of[Susceptance prices for the 'from bus' lines connected to bus *n* times the Susceptance value for that line]
- Interpretation 2: What if you had a single, radial line? Then $S_k = 0$

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Dual of the DCOPF: Identifying Terms

Dual of this particular DCOPF

Primal Problem

Primal Problem:

Minimize: $\sum_{g} c_g P_g$

$s.t.$

$$\sum_{\forall k(n,.)} P_k - \sum_{\forall k(.,n)} P_k + \sum_{\forall g(n)} P_g = d_n, \forall n \qquad (LMP_n)$$

$$-P_k \geq -P_k^{\max}, \forall k \qquad (F^{+}_{k})$$

$$P_k \geq -P_k^{\max}, \forall k \qquad (F^{-}_{k})$$

$$B_k(\theta_n - \theta_m) - P_k = 0, \forall k \qquad (S_k)$$

$$-P_g \geq -P_g^{\max}, \forall g \qquad (\alpha_g)$$

$P_g \geq 0$; θ_n, P_k free

Complete Dual Formulation

Dual Problem:

Maximize: $\sum_{n} d_n LMP_n - \sum_{k} P_k^{\max}\left(F_k^{+} + F_k^{-}\right) - \sum_{g} P_g^{\max}\alpha_g$

$s.t.$

$$LMP_n - LMP_m - F_k^{+} + F_k^{-} - S_k = 0, \forall k \qquad (P_k)$$

$$LMP_n - \alpha_g \le c_g, \forall g \qquad (P_g)$$

$$\sum_{k(n,.)} B_k S_k - \sum_{k(.,n)} B_k S_k = 0, \forall n \qquad (\theta_n)$$

$F^{+}{}_k$, $F^{-}{}_k$, $\alpha_g \ge 0$; LMP_n, S_k free

Load Payment

Load Payment

- Load Payment:
  - Using a nodal pricing system (LMPs)
  - Each load pays its LMP
- Total Load Payment:
  - Sum of: (Load at Bus *n*) * (LMP_n)
  - Easily identifiable from the dual objective:

$$\sum_n d_n LMP_n$$

Generation Rent

Definition of Generation Rent

- Short-term generation profit (total profit among all generators)
- Difference between generation revenue and generation cost
- First, Generator *g*'s Revenue:
  - $LMP_n P_g$
- Second, Generator *g*'s Cost:
  - $c_g P_g$
- Generator *g*'s Rent (short term profit):
  - $(LMP_n - c_g)P_g$

Identification of Generation Rent

- Complementary Slackness tells us that, at optimality:

$$\left(LMP_n - \alpha_g\right)P_g = c_g P_g, \forall g \longrightarrow \alpha_g P_g = \left(LMP_n - c_g\right)P_g$$

$$P_g \alpha_g = P_g^{\max} \alpha_g, \forall g$$

- Therefore, the Total Generation Rent:

$$\sum_g P_g^{\max} \alpha_g$$

  - This is the last term in the Dual Objective

Congestion

Terminology

- Note that there are various terms used
- **Cost of congestion:** the cost to operate the grid with infinite transmission capacity (no congestion) versus actual cost
  - Also stated as the "congestion cost" but I prefer to list it as "cost of congestion" to be clearer on its meaning
- **Congestion rent:** Excess funds collected by the ISO (total load payment minus total generation revenue) caused by congestion (used to fund FTRs)
- Next section covers congestion rent

Congestion Rent

- Congestion Rent on a specific line:
- Flowgate marginal price * Line's flow
- Define:
  - $P_k(F^+_k - F^-_k)$
- Explanation:
  - If $P_k = P_k^{max}$, we have: $P_k F^+_k$ (which is non-negative)
  - If $P_k = P_k^{min} = -P_k^{max}$, we have: $-P_k F^-_k$ (which is non-negative as well)
  - **Again, remember that this is based on the particular DCOPF formulation I am using**

Congestion Rent

- Total system-wide Congestion Rent:

$$\sum_k P_k \left(F_k^+ - F_k^-\right)$$

- CS tells us: $$-P_k F_k^+ = -P_k^{\max} F_k^+$$

$$P_k F_k^- = -P_k^{\max} F_k^-$$

- So: $$\sum_k P_k \left(F_k^+ - F_k^-\right) = \sum_k P_k^{\max} \left(F_k^+ + F_k^-\right)$$
  - This is the Second term in the Dual's Objective

Break

Slides taken from:

EEE 598
Electric Energy Markets

School of ECEE, ASU

Professor Kory W. Hedman

Duality and
the Dual of the DCOPF

Important Note

- There are many different ways to formulate a DCOPF Problem
- Today, we will examine one particular formulation and its corresponding dual
- We will discuss the economic interpretation of this dual problem, its variables, and its constraints
- However, it is important to understand that the interpretations we develop apply to this DCOPF formulation
- If you take a dual of a different DCOPF formulation, the dual will not be the same and you may get different interpretations of that different dual

Dual of the DCOPF: Exchange of Money Identity

Dual of this particular DCOPF

Complete Dual Formulation

Dual Problem:

Maximize: $\sum_n d_n LMP_n - \sum_k P_k^{\max}\left(F_k^+ + F_k^-\right) - \sum_g P_g^{\max}\alpha_g$

s.t.

$$LMP_n - LMP_m - F_k^+ + F_k^- - S_k = 0, \forall k \qquad (P_k)$$

$$LMP_n - \alpha_g \le c_g, \forall g \qquad (P_g)$$

$$\sum_{k(n,.)} B_k S_k - \sum_{k(.,n)} B_k S_k = 0, \forall n \qquad (\theta_n)$$

$F^+{}_k$, $F^-{}_k$, $\alpha_g \ge 0$; LMP_n, S_k free

Identity

- By Strong Duality, we know that at optimality the primal objective = dual objective

$$\sum_g c_g P_g$$

$$= \sum_n d_n LMP_n - \sum_k P_k^{\max}\left(F_k^+ + F_k^-\right) - \sum_g P_g^{\max} \alpha_g$$

Identity

- By Strong Duality, we know that at optimality the primal objective = dual objective

$$\sum_g c_g P_g = \boxed{\sum_n d_n LMP_n} - \sum_k P_k^{\max}\left(F_k^+ + F_k^-\right) - \boxed{\sum_g P_g^{\max}\alpha_g}$$

- Generation Cost
- Load Payment
- Congestion Rent
- Generation Rent

Exchange of Money

- **Identity:**

Load Payment $$\sum_n d_n LMP_n$$

= Generation Revenue $$= \left\{ \sum_g c_g P_g + \sum_g P_g^{\max} \alpha_g \right.$$

+ Congestion Rent $$\left. + \sum_k P_k^{\max} \left(F_k^{+} + F_k^{-} \right) \right\}$$

Break

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Duality and
the Dual of the DCOPF

Important Note

- There are many different ways to formulate a DCOPF Problem
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Dual of the DCOPF: Alternative Definitions for Congestion Rent

Dual of this particular DCOPF

Alternative Definition Given for Congestion Rent

- More common definition given for congestion rent:
- Difference between the load payment and the generation revenue

- This is the same as what we derived and proved:
  Load Payment = Generation Revenue + Congestion Rent

Alternative Definition Given for Congestion Rent

- Another definition given for congestion rent:
- Sum of the Cost to send *P* MWs from bus *m* to bus *n* (for all transactions)

  $\sum_{\boldsymbol{k}} P_k{*}(LMP_n - LMP_m)$
- Is this the same as what we developed?
- Let's treat it this way: cost to send P_k between the buses it is connected to, bus *m* and bus *n*
- So it is:

  $\boldsymbol{\sum_k [P_k(LMP_n - LMP_m)]}$

$$\sum_k P_k\left(LMP_n - LMP_m\right)$$

Alternative Definition Given for Congestion Rent

$$\sum_k P_k \left(LMP_n - LMP_m \right)$$

- This is often seen as another way to calculate the system wide congestion rent
- Does it equal our previous definition?

$$\sum_k P_k \left(F_k^+ - F_k^- \right) ???$$

Alternative Definition Given for Congestion Rent

- Take the following constraint from the dual problem:

$$\sum_{k(n,.)} B_k S_k - \sum_{k(.,n)} B_k S_k = 0, \forall n \qquad (\theta_n)$$

- With CS, we have: $\sum_{k(n,.)} B_k \theta_n S_k - \sum_{k(.,n)} B_k \theta_n S_k = 0$

- Now, sum over all *n*:

$$\sum_{n} \left[\sum_{k(n,.)} B_k \theta_n S_k - \sum_{k(.,n)} B_k \theta_n S_k \right] = 0$$

Alternative Definition Given for Congestion Rent

- Take the previous equation:

$$\sum_{n}\left[\sum_{k(n,.)} B_k \theta_n S_k - \sum_{k(.,n)} B_k \theta_n S_k\right] = 0$$

- Can be rewritten as:

$$\sum_{k} B_k \left(\theta_n - \theta_m\right) S_k = 0 = \sum_{k} P_k S_k$$

Alternative Definition Given for Congestion Rent

- Next, take this equation from the dual:

$$LMP_n - LMP_m - F_k^+ + F_k^- - S_k = 0, \forall k \qquad \left(P_k\right)$$

- With CS, we have:

$$P_k\left(LMP_n - LMP_m - F_k^+ + F_k^- - S_k\right) = 0$$

- Sum over all *k*:

$$\sum_k P_k\left(LMP_n - LMP_m - F_k^+ + F_k^- - S_k\right) = 0$$

Alternative Definition Given for Congestion Rent

- So we have: $\sum_k P_k \left(LMP_n - LMP_m - F_k^{+} + F_k^{-} - S_k \right) = 0$
- And we know that: $\sum_k B_k \left(\theta_n - \theta_m \right) S_k = 0 = \sum_k P_k S_k$
- This gives:

$$\sum_k P_k \left(LMP_n - LMP_m \right) = \sum_k P_k \left(F_k^{+} - F_k^{-} \right)$$

- **Proves that the total congestion rent = LMP difference for each line * Line Flow**