Module 6

Grid Security

How did the yesterday's definition differ from tomorrow's?

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Arizona State University

Module 6.1

Traditional Security Framework

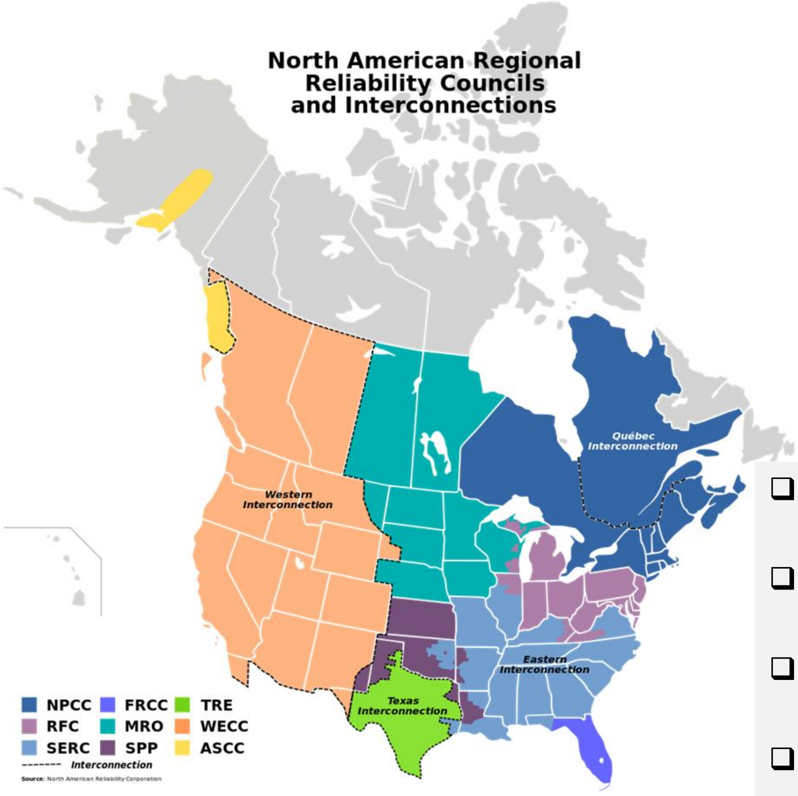
What is power system security?

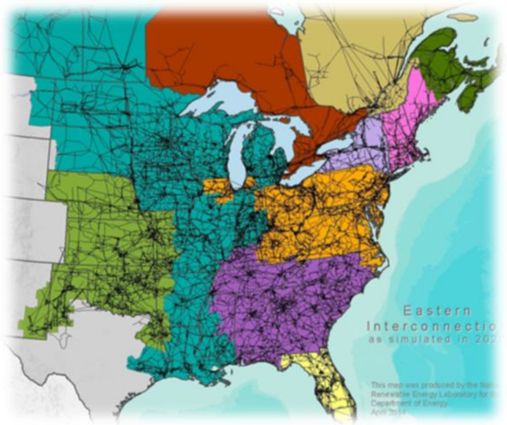
Arizona State University

Traditional Security Framework

1. Control center framework
2. The role of applications
3. Software system and redundancy
4. Digital protective relaying

Power Grid Interconnections





- More than 100 control areas
- Tens of thousands of substations
- Situational awareness
- Possibilities of cyberattack on IP-based substations

Power Control Centers

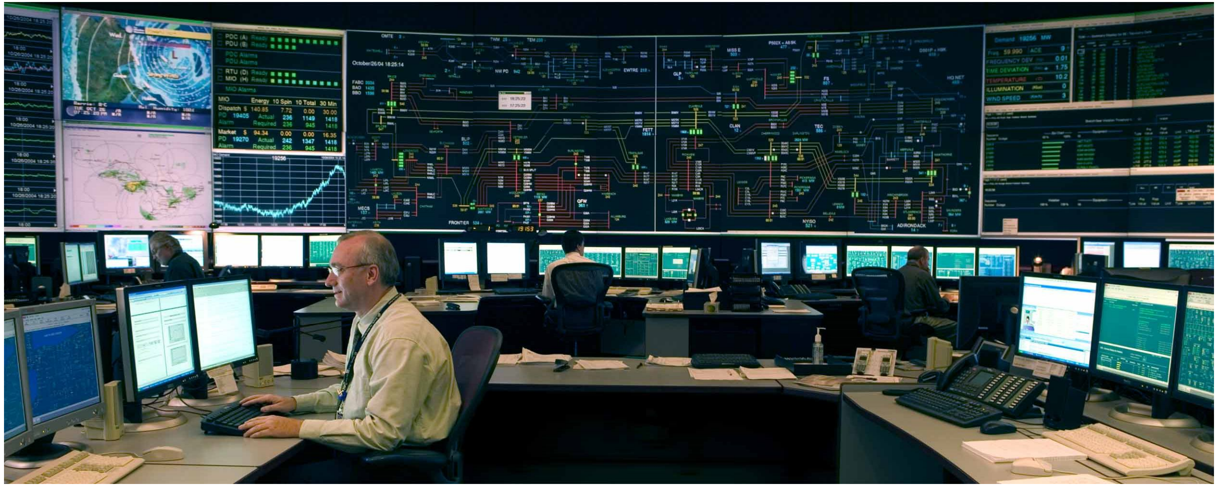


Photo Courtesy: http://www.temetprotection.com.ar/temet_advanced_shelter_control_system.html

- ❑ Supervisory Control and Data Acquisition (SCADA)
  - ❑ Data acquisition – analog (P, Q, V, etc.) and digital (switch status) measurements
  - ❑ **Alarms are derived** from these measurements over given time
  - ❑ Preventive and remedial controls

- ❑ Complete information of the physical health of a power system under a utility's grid territory

Revolution of SCADA System

Evolved through 3 generations

1. First Generation (Monolithic)
2. Second Generation (Distributed)
3. Third Generation (Networked)

First Generation (Monolithic)

1. First developed system centered in Mainframe (all-in-one role)

2. Stand alone

3. Communication protocols to RTUs are proprietary (vendor-dependent)

4. Redundancy and connectivity very much depends on the features of a vendor

Second Generation (Distributed)

1. Implemented using Local Area Network (LAN) technology
2. Distribute the computation burdens to the number of computers via LAN
3. Servers include communication, application, user interface, database
4. Redundancy between servers provide fail-over capability
5. Limitation of hardware and software provided by vendors

Third Generation (Networked)

1. Close to second generation framework

2. More open system environment rather than proprietary. Standardized protocols such as DNP3.0i, IEC61850, Modbus, RS-232, etc.

3. Implementation of WAN communication, e.g., Internet Protocol (IP)
    1. Disaster survivability for multi-site failover
    2. Wide area monitoring, control, and protection

4. Cost effective as hardware and software are PC-based

5. The use of standardized WAN communication derives cybersecurity problems

IP-Based Distributed SCADA

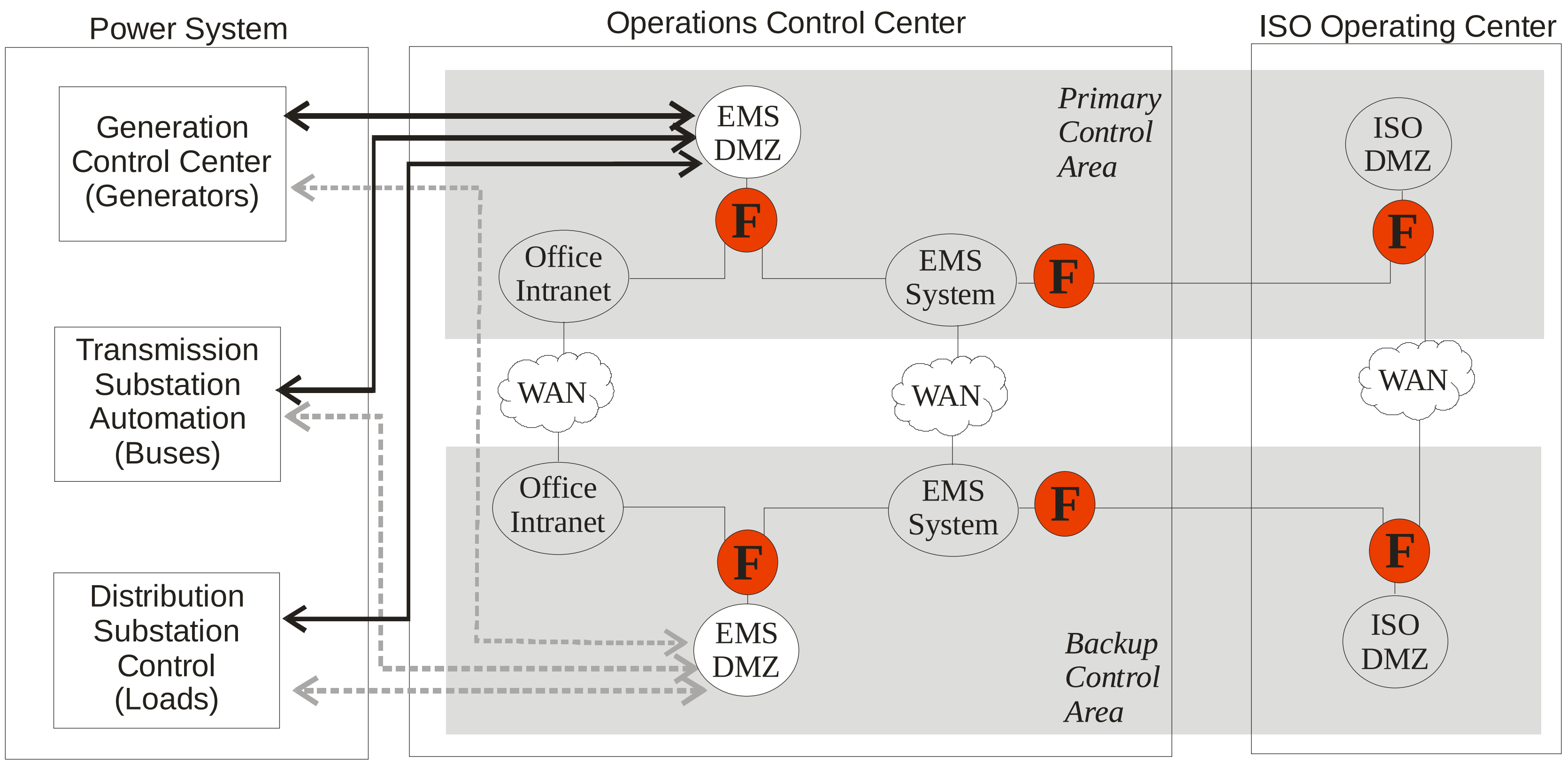
1. Current trend in the development of distributed control centers

2. Do not depend on fixed dedicated communication from RTUs to Control Center

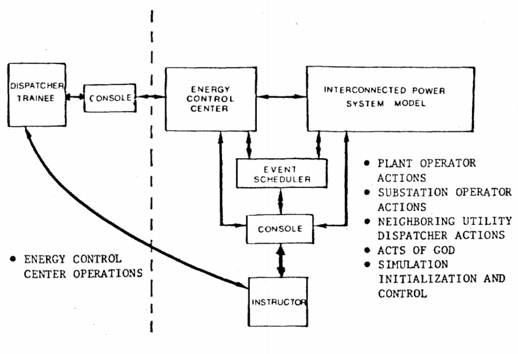
3. Standard protocol enables the use of heterogeneous components

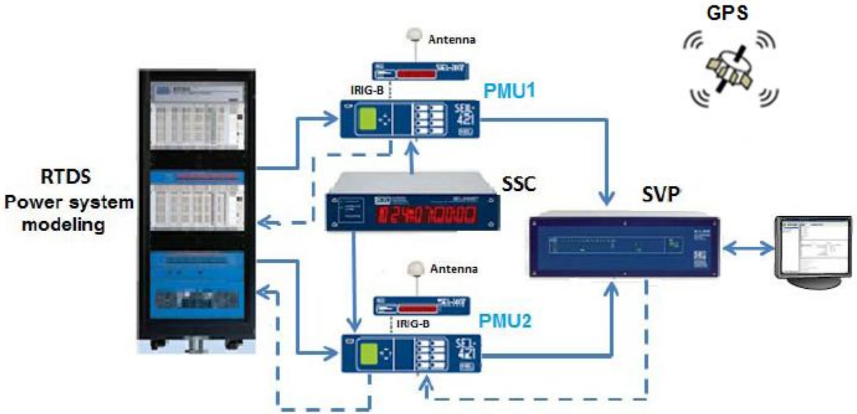
4. Speed vs. Cybersecurity

Example Control Center Network Interconnectivity



Dispatcher Training Simulators (DTS) vs. Real-Time Digital Simulator (RTDS) to **Faster Than Real Time**. Then **Digital Twin**





- ❑ Computing advancement (both software and hardware performance)
- ❑ Higher data resolutions
- ❑ Mimic more real-world environment with dynamic models (power flow is a steady-state model)
- ❑ PMU, RTU, legacy system, fault diagnosis
- ❑ Training operators / dispatchers to be more decisive based on information observed.

Electrical Short Circuit
Why we need power system protection?

Short circuit that everyone can relate ☺

These arcs will destroy <u>tens of millions</u> of assets in substations if there is no circuit breakers in substations

Tens of Thousands of Electrical Substations

- High voltage (200kV, 345 kV, 500kV, 765kV…)
- Complex process to commissioning a substation and automation
- Stringent compliance and long process

Substation Switchgear and Automation

Circuit Breaker

Substation Computer

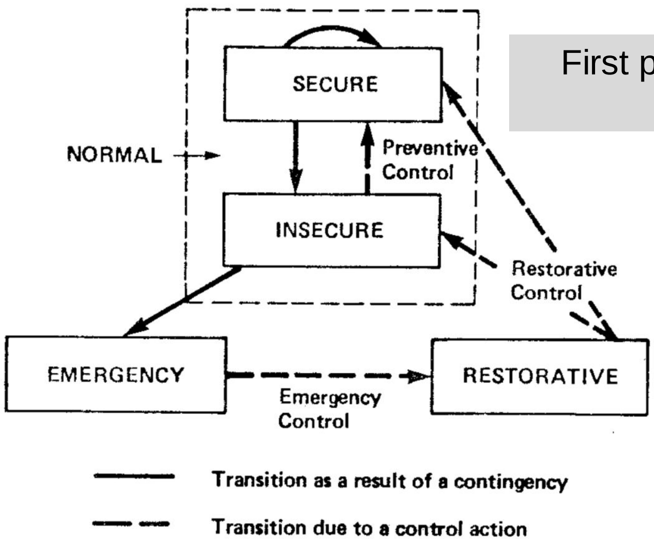
Transmission lines

Disconnector

Digital Relay

- ❑ 3-phase circuits
- ❑ Transmission/distribution utilities
- ❑ Long distance power transfer
- ❑ Long process to commission a substation
- ❑ Research thrusts: Power Systems Modeling, Power Delivery and Automation

Security State Transitions

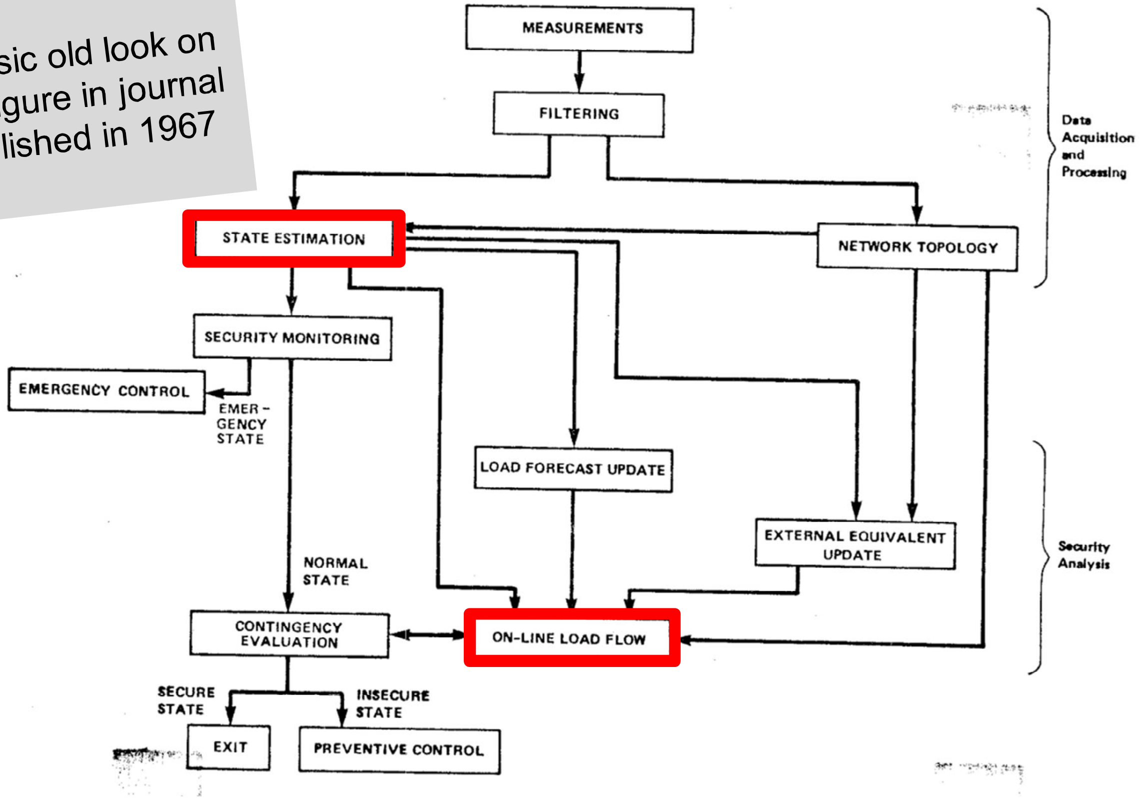


First published in 1967

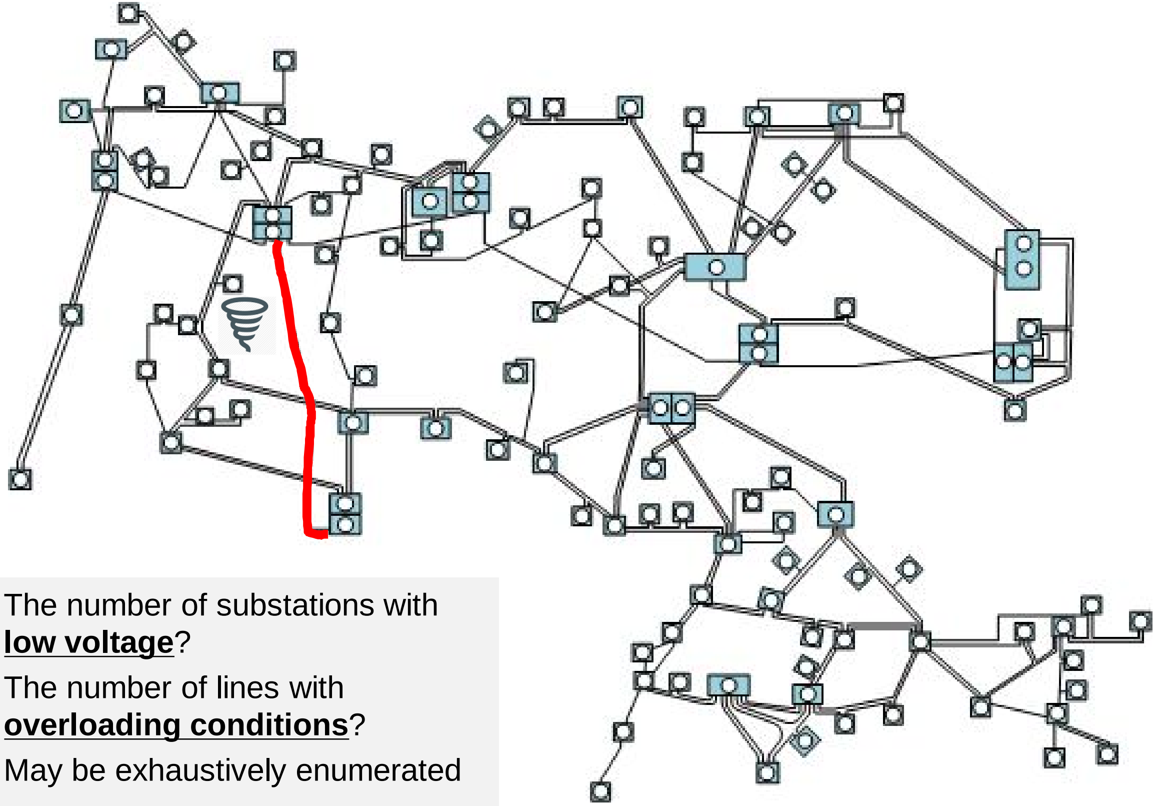
- Power system operating states and the associated state transitions due to CONTINGENCIES and CONTROL functions
- Thomas E. Dy Liacco, “The Adaptive Reliability Control System,” *IEEE Transactions on Power Apparatus and Systems*, Vol. PAS-86, No. 5, May 1967.

Centralized Security Control Functions

Classic old look on the figure in journal published in 1967

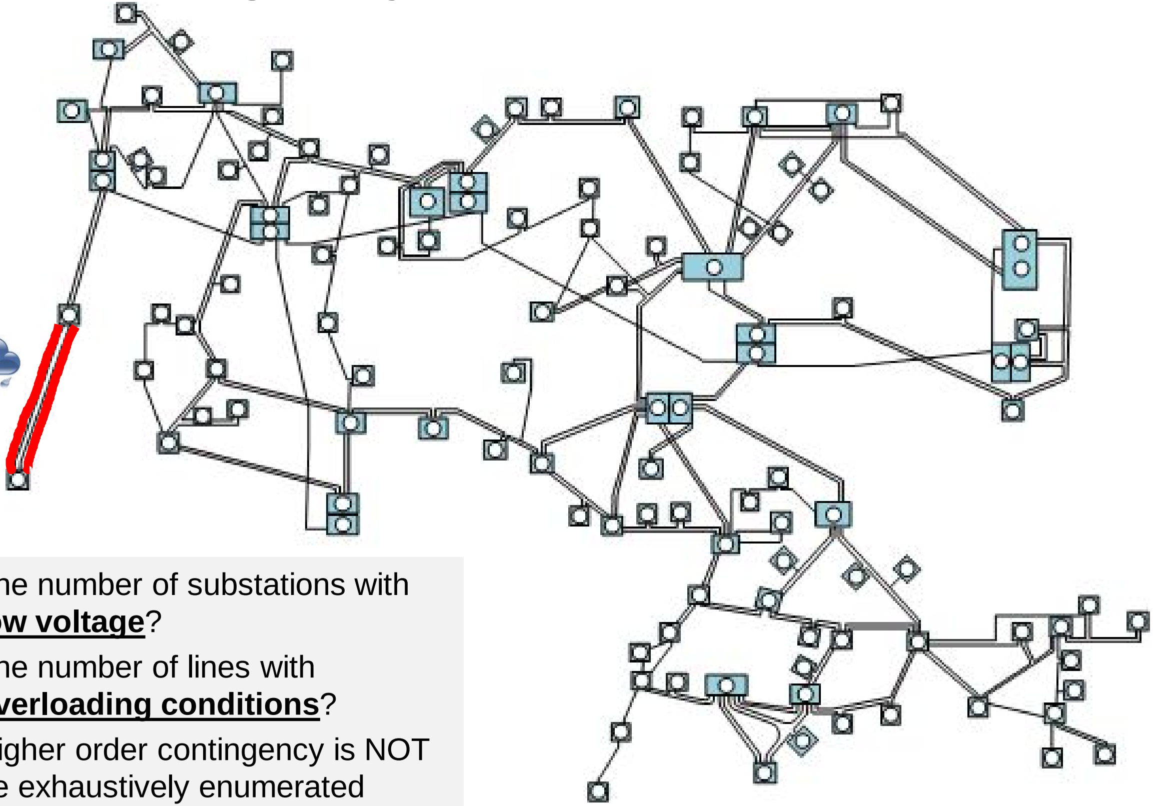


IEEE 118-bus System:
Traditional Power System **Security**: N-1 Contingency



- ❑ The number of substations with **low voltage**?
- ❑ The number of lines with **overloading conditions**?
- ❑ May be exhaustively enumerated

N-2 Contingency

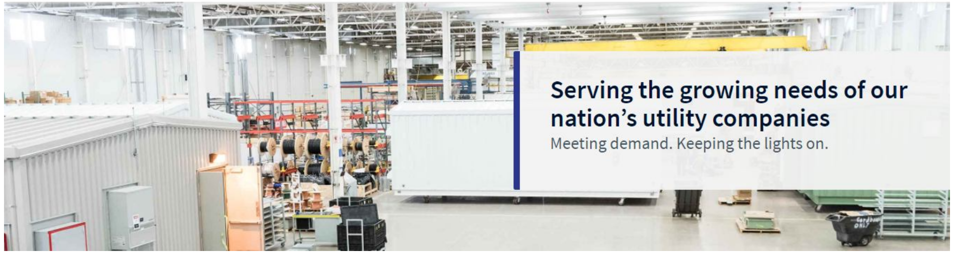


- The number of substations with **low voltage**?
- The number of lines with **overloading conditions**?
- Higher order contingency is NOT be exhaustively enumerated

Our Community's Industry Constituent

SYSTEMS CONTROL

TURNKEY SOLUTIONS | EQUIPMENT ENCLOSURES | CONTROL & RELAY PANELS | ENGINEERING SERVICES



EQUIPMENT ENCLOSURES

We have the capabilities to design Equipment Enclosures to your unique specifications for a truly customized solution.

CONTROL & RELAY PANELS

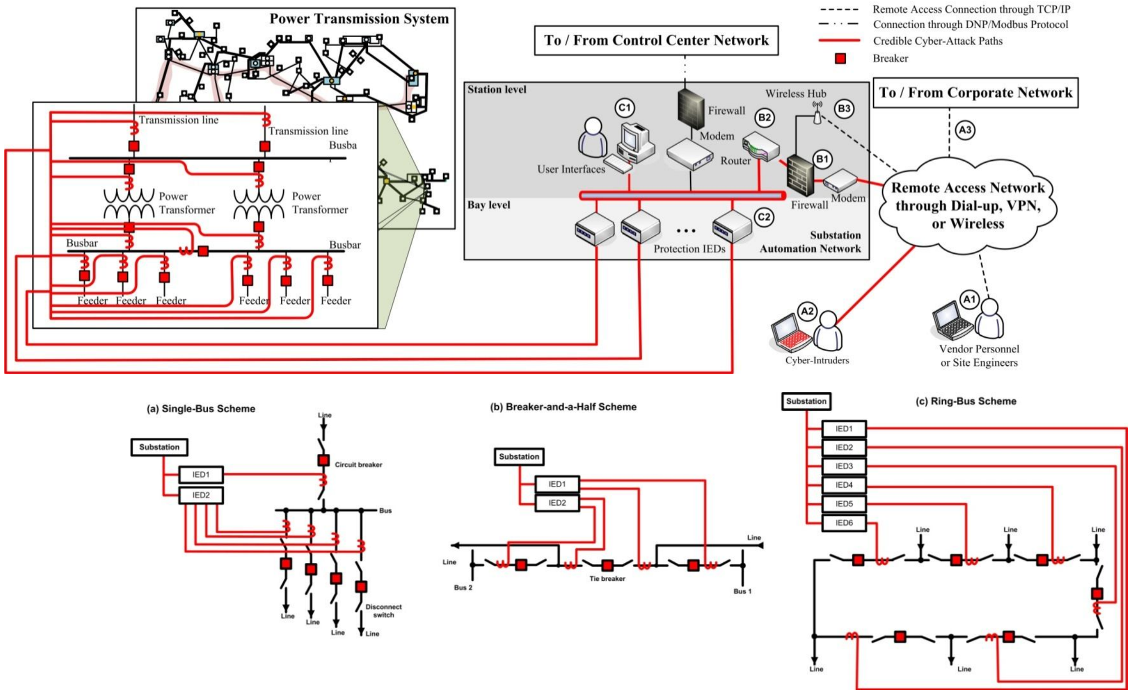
At Systems Control, our control and relay panels are custom-designed and built to meet your exact specifications.

ENGINEERING SERVICES

Our design and engineering team is designed to supplement your core engineering team across all disciplines.

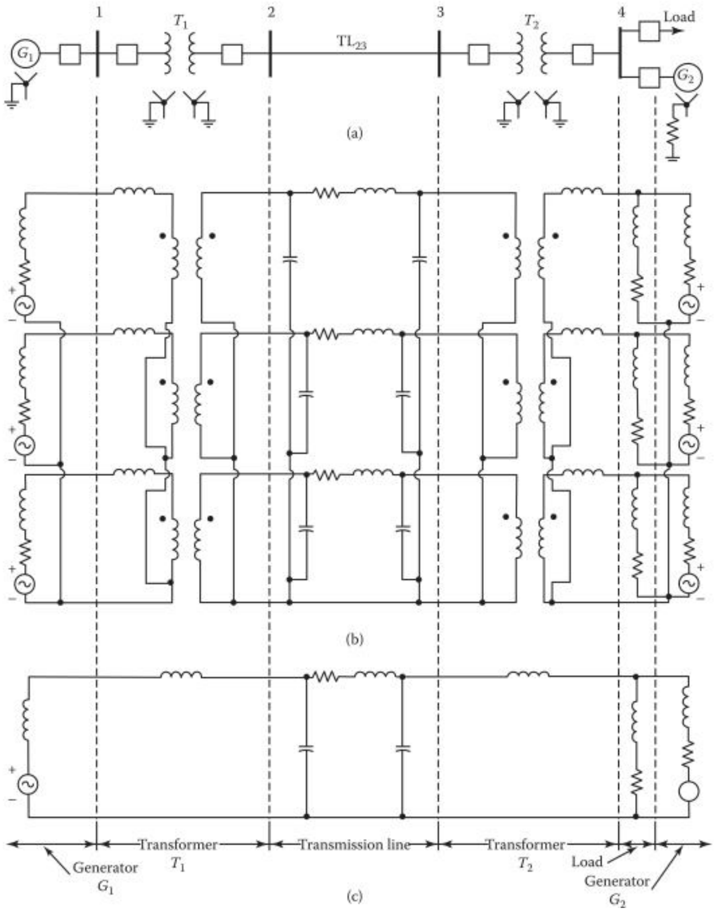
- ❑ Substation automation and solutions
- ❑ Tradition to hire our engineering students
- ❑ Tens of thousands of Substations in North America will transition to IP-based substations
- ❑ Templates, sandboxes for security architecture systems

Exploring the Details of Substation Topology



- Permutations of switching sequence that will cause maximum damage to system instability.

Graphical Representations

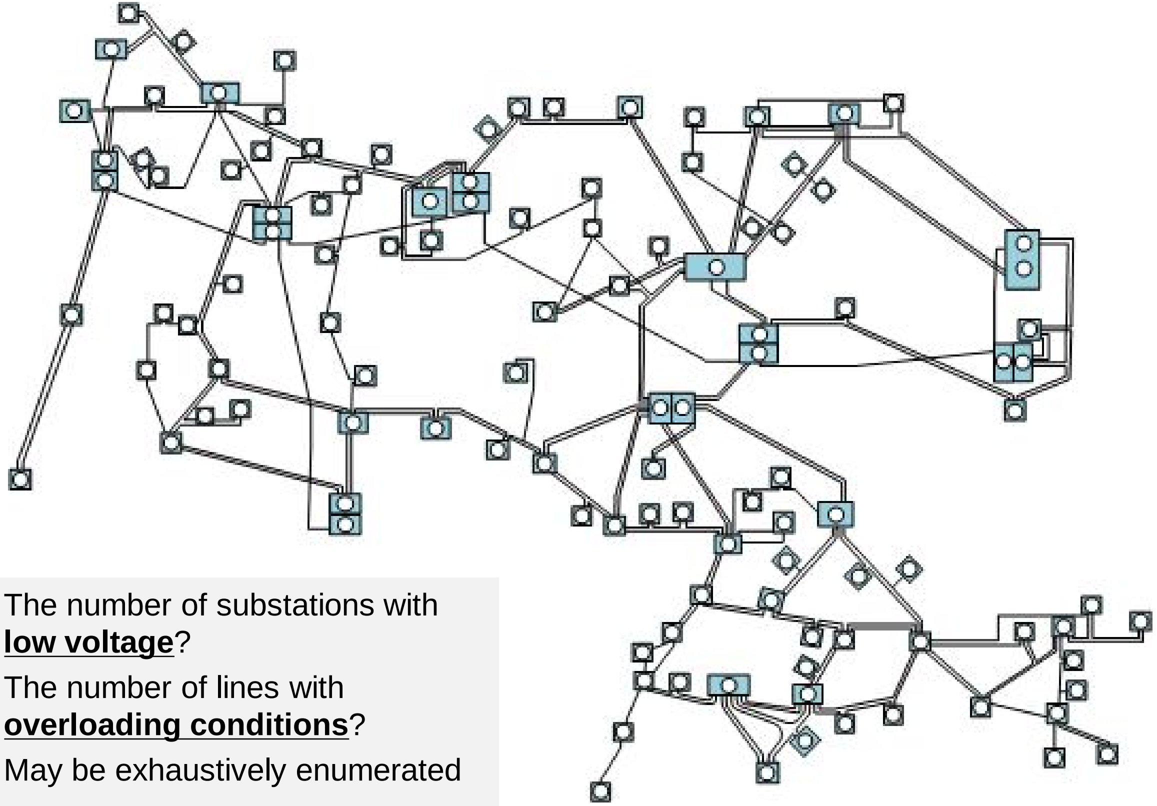


One-line diagram

3-phase representation in electrical circuit

Simplified one-phase representation in electrical circuit

Base case status: Voltages at all buses and power flow for all branches?



- The number of substations with **low voltage**?
- The number of lines with **overloading conditions**?
- May be exhaustively enumerated

Power Flow Application and Basics

- New generation sites
- New transmission line locations
- Which new DC links provide the best energy flows between areas?
- Which new FACT devices will provide the necessary control to maintain the
- system in a secure operating system?
- Where can new demand locations be added given the present system design?

Between Linear Circuit and Non-Linearity of a Power System

$$V = I \times R$$

- DC Circuit.
- Real. Not imaginary component

- AC Circuit.
- Real and imaginary parts of the complex power *S*

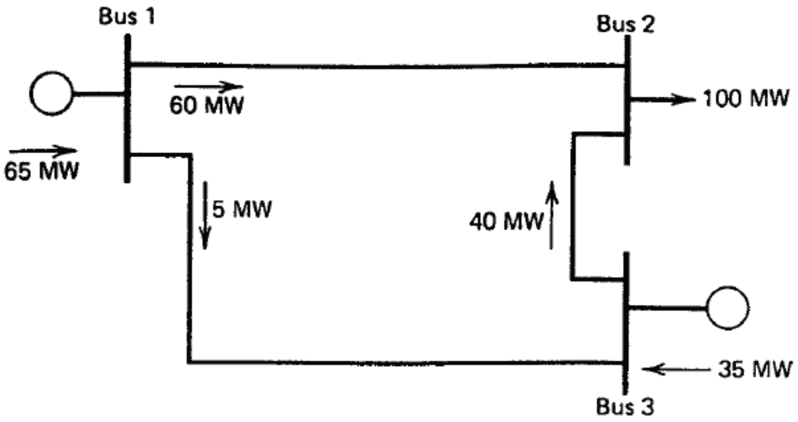
$$S = VI^{*} = P + jQ$$

$$S = P + Qi$$

In variable conflict with current, *i, in electrical engineering*

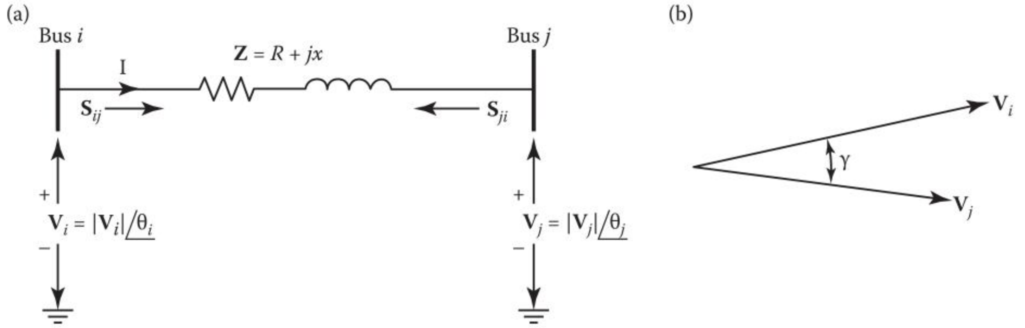
Bus Types of Power Flow Formulation

Bus Type	Known Quantities	Unknown Quantities
Slack	$\lvert V \rvert = 1.0$	P, Q
	$\theta = 0$	
Generator (PV bus)	$P, \lvert V \rvert$	Q, θ
Load (PQ bus)	P, Q	$\lvert V \rvert, \theta$



- Bus is often referred to node in a network

Per-Phase Representation of a Transmission Line



$$\mathbf{I} = \frac{\mathbf{V}_i - \mathbf{V}_j}{Z}$$

$$\mathbf{S}_{ij} = \mathbf{V}_i \left(\frac{\mathbf{V}_i^* - \mathbf{V}_j^*}{Z} \right)$$
$$= \frac{|\mathbf{V}_i|^2 - |\mathbf{V}_i||\mathbf{V}_j|\angle\theta_i - \theta_j}{R - jX}$$

$$[\mathbf{V}_{\text{bus}}] = [\mathbf{Y}_{\text{bus}}]^{-1}[\mathbf{I}_{\text{bus}}]$$
$$= [\mathbf{Z}_{\text{bus}}][\mathbf{I}_{\text{bus}}]$$

Embedded with topological connectivity

$$[\mathbf{I}_{\text{bus}}] = [\mathbf{Y}_{\text{bus}}][\mathbf{V}_{\text{bus}}]$$

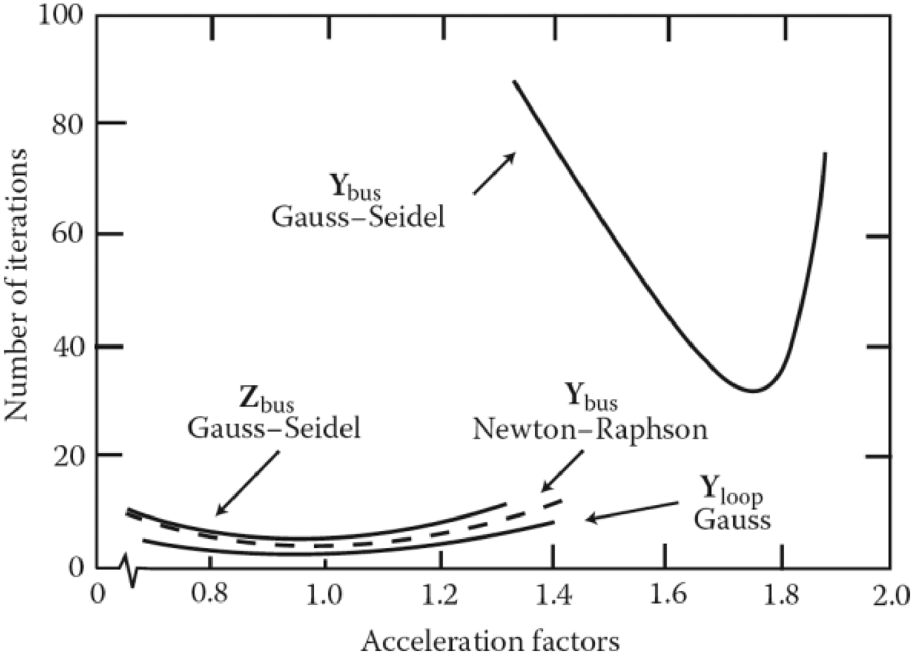
- A “bus” is often referred to a node in a network

Power Flow Methods

- It is an <u>iterative</u> process to seek for “convergence”
  1. Gauss-Seidal
  2. Newton-Raphson
  3. Fast Decoupled

- A converged power flow solution implies the system has met the criteria where power at all nodes sum up to be zero.

Effects of acceleration factors on rate of convergence for power flow solutions.



Effects of acceleration factors on rate of convergence for power flow solutions. (From Stagg, G. W., and El-Abiad, A. H., Computer Methods in Power System Analysis. McGraw-Hill, New York, 1968.)

- ❑ Comparison between iterations and acceleration factors

$$V_{i(\text{acceleration})^{(k+1)}} = \mathbf{V}_i^{(k)} + \alpha(\mathbf{V}_i^{(k+1)} - \mathbf{V}_i^{(k)})$$
$$= \mathbf{V}_i^{(k)} + \alpha \cdot \Delta\mathbf{V}_i^{(k)}$$

Gauss-Seidal Method

$$\mathbf{V}_1^{(k+1)} = \frac{1}{\mathbf{Y}_{11}}(\mathbf{I}_1^{(k)} - \mathbf{Y}_{12}\mathbf{V}_2^{(k)} - \mathbf{Y}_{13}\mathbf{V}_3^{(k)} - \cdots - \mathbf{Y}_{1n}\mathbf{V}_n^{(k)})$$

$$\mathbf{V}_2^{(k+1)} = \frac{1}{\mathbf{Y}_{22}}(\mathbf{I}_2^{(k)} - \mathbf{Y}_{21}\mathbf{V}_2^{(k+1)} - \mathbf{Y}_{23}\mathbf{V}_3^{(k)} - \cdots - \mathbf{Y}_{2n}\mathbf{V}_n^{(k)})$$

$$\mathbf{V}_3^{(k+1)} = \frac{1}{\mathbf{Y}_{33}}(\mathbf{I}_3^{(k)} - \mathbf{Y}_{31}\mathbf{V}_2^{(k+1)} - \mathbf{Y}_{32}\mathbf{V}_3^{(k+1)} - \cdots - \mathbf{Y}_{3n}\mathbf{V}_n^{(k)})$$

$$\mathbf{V}_n^{(k+1)} = \frac{1}{\mathbf{Y}_{nn}}(\mathbf{I}_n^{(k)} - \mathbf{Y}_{n1}\mathbf{V}_2^{(k+1)} - \mathbf{Y}_{n2}\mathbf{V}_2^{(k+1)} - \cdots - \mathbf{Y}_{nn}\mathbf{V}_{n-1}^{(k+1)})$$

PQ-Bus

$$\mathbf{I}_i = \left(\frac{P_i + jQ_i}{\mathbf{V}_i}\right)^* = \frac{P_i - jQ_i}{\mathbf{V}_i^*}$$

$$\mathbf{V}_i^{(k+1)} = \frac{1}{\mathbf{Y}_{ii}}\left(\frac{P_i - jQ_i}{\mathbf{V}_i^{(k)*}} - \sum_{j=1}^{n}\mathbf{Y}_{ij}\mathbf{V}_{ij}^{(k)}\right) \quad \text{for } i = 2, \cdots, n$$

PV-Bus

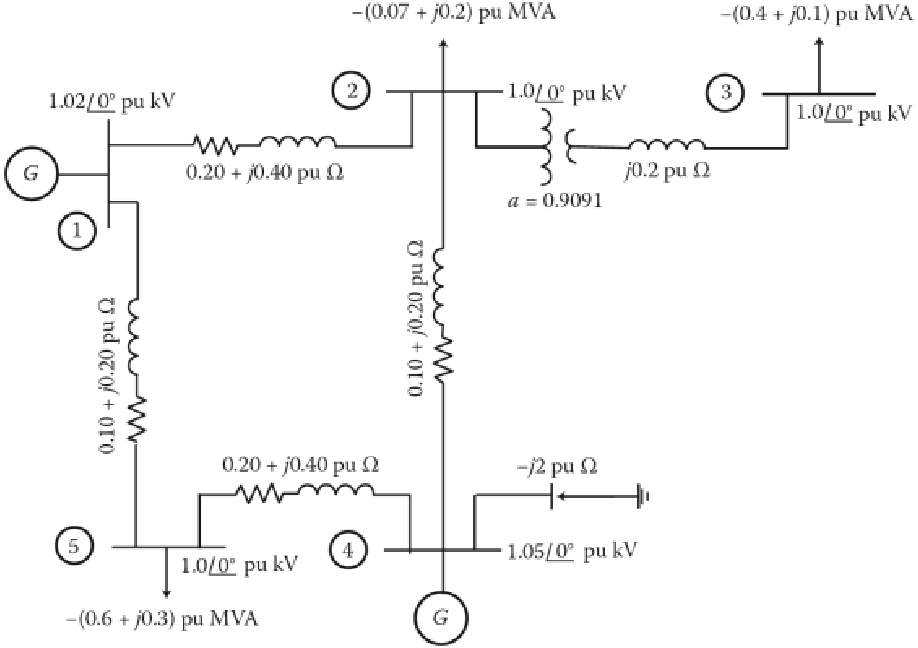
$$\mathbf{I}_{\text{gen}} = \frac{P_i - jQ_i}{\mathbf{V}_i^*} = \mathbf{Y}_{i1}\mathbf{V}_1 + \mathbf{Y}_{i2}\mathbf{V}_2 + \mathbf{Y}_{i3}\mathbf{V}_3 + \cdots + \mathbf{Y}_{in}\mathbf{V}_n$$

$$P_i - jQ_i = \mathbf{V}_i^{(k)}\left[\sum_{j=1}^{n}\mathbf{Y}_{ij}\mathbf{V}_j^{(k)}\right]$$

Admittance Matrix, Y

- Done by inspection
- Adjacency matrix + degree matrix + impedance info
- Topological connectivity
  - Substation (node/bus)
  - Shunt
  - Loads
  - Generators
- Electrical short circuit analysis applications
- Highly sparse matrix if size gets really big

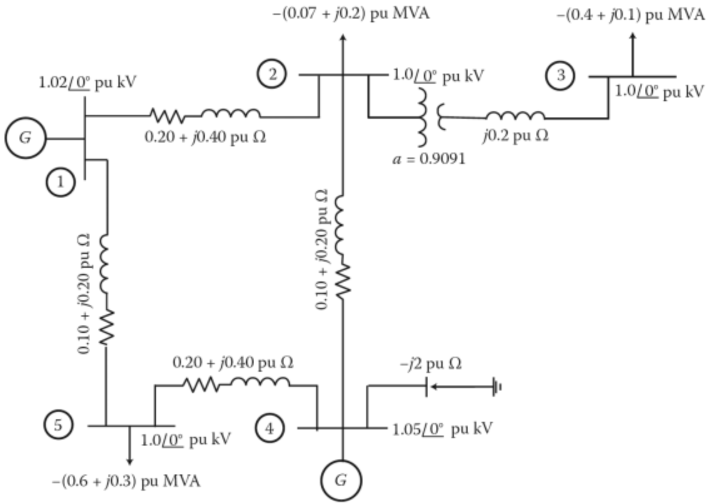
Admittance Matrix, Y



$$[\mathbf{Y}_{\text{bus}}] = \begin{bmatrix} \mathbf{Y}_{11} & \mathbf{Y}_{12} & \mathbf{Y}_{13} & \mathbf{Y}_{14} & \mathbf{Y}_{15} \\ \mathbf{Y}_{21} & \mathbf{Y}_{22} & \mathbf{Y}_{23} & \mathbf{Y}_{24} & \mathbf{Y}_{25} \\ \mathbf{Y}_{31} & \mathbf{Y}_{32} & \mathbf{Y}_{33} & \mathbf{Y}_{34} & \mathbf{Y}_{35} \\ \mathbf{Y}_{41} & \mathbf{Y}_{42} & \mathbf{Y}_{43} & \mathbf{Y}_{44} & \mathbf{Y}_{45} \\ \mathbf{Y}_{51} & \mathbf{Y}_{52} & \mathbf{Y}_{53} & \mathbf{Y}_{54} & \mathbf{Y}_{55} \end{bmatrix}$$

$$= \begin{bmatrix} (3-j6) & (-1+j2) & (0+j0) & (0+j0) & (-2+j4) \\ (-1+j2) & (3-j12.05) & (0+j5.5) & (-2+j4) & (0+j0) \\ (0+j0) & (0+j5.5) & (0-j5) & (0+j0) & (0+j0) \\ (0+j0) & (-2+j4) & (0+j0) & (3-j5.5) & (-1+j2) \\ (-2+j4) & (0+j0) & (0+j0) & (-1+j2) & (3-j6) \end{bmatrix}$$

Gauss-Seidal Method Using Admittance Matrix, Y



$$\mathbf{V}_2^{(1)} = \frac{1}{\mathbf{Y}_{22}}\left(\frac{P_2 - jQ_2}{\mathbf{V}_2^{(0)*}} - \sum_{j=1_{j\neq 2}}^{5} \mathbf{Y}_{2j}\mathbf{V}_j^0\right)$$

$$= \frac{1}{\mathbf{Y}_{22}}\left(\frac{P_2 - jQ_2}{\mathbf{V}_2^{(0)*}} - \left\{\mathbf{V}_1^{(0)}\mathbf{Y}_{21} + \mathbf{V}_3^{(0)}\mathbf{Y}_{23} + \mathbf{V}_4^{(0)}\mathbf{Y}_{24} + \mathbf{V}_5^{(0)}\mathbf{Y}_{25}\right\}\right)$$

$$= \frac{1}{3 - j12.05}\left[\frac{-0.7 + j(0.2)}{1.0} - \{1.02(-1 + j2) + 1.0(0 + j5.5) + 1.05(-2 + j4) + 1.0(0 + j0)\}\right]$$

$$= \frac{1}{3 - j12.05}\left[\frac{-0.7 + j(0.2)}{1.0} - (-3.12 + j11.74)\right]$$

$$= \frac{1}{3 - j12.05}[2.42 - j11.54]$$

$$= 0.9494\angle -2.1366^\circ$$

$$= 0.94886 - j0.035 \text{ pu}$$

$$\mathbf{V}_2^{(2)} = \frac{1}{\mathbf{Y}_{22}}\left(\frac{P_2 - jQ_2}{\mathbf{V}_2^{(1)*}} - \left\{\mathbf{V}_1^{(0)}\mathbf{Y}_{21} + \mathbf{V}_3^{(0)}\mathbf{Y}_{23} + \mathbf{V}_4^{(0)}\mathbf{Y}_{24} + \mathbf{V}_5^{(0)}\mathbf{Y}_{25}\right\}\right)$$

$$= \frac{1}{3 - j12.05}\left[\frac{-0.7 + j(0.2)}{0.9495\angle 2.1366^\circ} - \{-3.12 + 11.74\}\right]$$

$$= \frac{1}{3 - j12.05}[2.3911 - j11.502]$$

$$\cong 0.946\angle -2.2364^\circ$$

$$\cong 0.94533 - j0.0369 \text{ pu}$$

Impedance Matrix, Z

- Inverse of admittance Y matrix
- <u>**CANNOT**</u> be done by inspection
- <u>**Individual element has different meaning**</u>**:** ***Equivalent circuit*** from the two nodes, e.g., z_{23} represents equivalent circuit seen from nodes 2 and 3
- Strong application of modification on electrical short circuit analysis applications as well as power flow
- Full rank matrix
- Management of matrix singularity issues

Newton-Raphson Method

Start from Taylor's Series

$$f(x) = f(x_0) + \frac{1}{1!}\frac{df(x_0)}{dx}(x - x_0) + \frac{1}{2!}\frac{df^2(x_0)}{dx^2}(x - x_0)^2 + \cdots + \frac{1}{n!}\frac{df^n(x_0)}{dx^n}(x - x_0)^n = 0$$

$$f(x) = f(x_0) + \frac{df(x_0)}{dx}(x - x_0) = 0 \longrightarrow x^{(1)} = x^{(0)} - \frac{f(x^{(0)})}{df(x^{(0)})/dx}$$

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$$

$$\Delta x = -\frac{f(x^{(k)})}{f'(x^{(k)})}$$

Reorganize

$$\rightarrow \Delta x = x^{k+1} - x^{(k)}$$

Newton-Raphson Method

$$f(x) = f(x_0) + \frac{df(x_0)}{dx}(x - x_0) = 0$$

→ Taylor's Series only first derivative

$$F(x) = F(x^{(x)}) + [j(x^0)][x - x^{(0)}] = 0$$

In Matrix format

$$[x^{(k+1)}] = [x^{(k)}] - [J(x^{(x)})]^{-1}[F(x^k)]$$

$$[J(x)] \triangleq \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

$$\begin{aligned} [\Delta x] &= [x^{k+1}] - [x^{(k)}] \\ &= [J(x^{(x)})]^{-1}[F(x^k)] \end{aligned}$$

Example: Newton-Raphson Method

Determine the values of x_1 and x_2 by using the Newton–Raphson method.

$$f_1(x_1, x_2) = x_1^2 + 3x_1x_2 - 4 = 0$$

$$f_2(x_1, x_2) = x_1x_2 - 2x_2^2 + 5 = 0$$

Example: Newton-Raphson Method

Determine the values of x_1 and x_2 by using the Newton–Raphson method.

$$f_1(x_1, x_2) = x_1^2 + 3x_1x_2 - 4 = 0$$
$$f_2(x_1, x_2) = x_1x_2 - 2x_2^2 + 5 = 0$$

The Jacobian matrix is

$$[J(x)] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 & 3x_1 \\ x_2 & x_1 - 3x_2 \end{bmatrix}$$

As the initial approximation, let

$$[x^{(0)}] = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore, the first iteration can be performed as

$$[x^{(1)}] = [x^{(0)}] - \begin{bmatrix} 8 & 3 \\ 2 & -7 \end{bmatrix}^{-1} [f(x^{(0)})] = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} \frac{7}{62} & \frac{3}{62} \\ \frac{2}{62} & -\frac{8}{62} \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.7741 \\ 1.7742 \end{bmatrix}$$

Thus, after substituting the new values for the second iteration,

$$[x^{(2)}] = \begin{bmatrix} 0.7097 \\ 1.7742 \end{bmatrix} - \begin{bmatrix} 6.7419 & 2.1290 \\ 1.7742 & -6.3871 \end{bmatrix}^{-1} \begin{bmatrix} 0.2810 \\ -0.0364 \end{bmatrix} = \begin{bmatrix} 0.6730 \\ 1.7583 \end{bmatrix}$$

Example: Newton-Raphson Method

Determine the values of x_1 and x_2 by using the Newton–Raphson method.

$$f_1(x_1, x_2) = x_1^2 + 3x_1x_2 - 4 = 0$$
$$f_2(x_1, x_2) = x_1x_2 - 2x_2^2 + 5 = 0$$

Similarly, after substituting the new values for the third iteration,

$$\left[x^{(3)}\right] = \begin{bmatrix} 0.6730 \\ 1.7583 \end{bmatrix} - \begin{bmatrix} 6.6210 & 2.0191 \\ 1.7383 & -6.3602 \end{bmatrix}^{-1} \begin{bmatrix} 0.0031 \\ 0.0001 \end{bmatrix} = \begin{bmatrix} 0.6726 \\ 1.7582 \end{bmatrix}$$

Finally, after substituting for the last iteration,

$$\left[x^{(4)}\right] = \begin{bmatrix} 0.6726 \\ 1.7582 \end{bmatrix} - \begin{bmatrix} 6.6198 & 2.0178 \\ 1.7582 & -6.3602 \end{bmatrix} = \begin{bmatrix} 0.6726 \\ 1.7582 \end{bmatrix}$$

Therefore, it is obvious that the iterations have rapidly converged toward the results of $x_1^{(4)}$ = 0.6726 and $x_2^{(4)}$ = 1.7582.

Newton-Raphson Method for Power Flow Formulation

$$\mathbf{S}_i = P_i - jQ_i = \mathbf{V}_i^* \mathbf{I}_i$$

Rectangular Form

Polar Form

$$V_i \triangleq |\mathbf{V}_i| \angle \delta_i$$

$$V_i \triangleq |\mathbf{V}_i| \angle -\theta_{ij}$$

$$\mathbf{S}_i = P_i - jQ_i = \mathbf{V}_i^* \sum_{j=1}^{n} \mathbf{Y}_{ij} \mathbf{V}_j$$

$$P_i - jQ_i = (e_i - jf_i) \sum_{j=1}^{n} (G_{ij} - jB_{ij})(e_j + jf_j)$$

$$\mathbf{I}_i = \sum_{j=1}^{n} \mathbf{Y}_{ij} \mathbf{V}_j = \sum_{j=1}^{n} |\mathbf{Y}_{ij}||\mathbf{V}_j| \angle -\theta_{ij} + \delta_j$$

$$\mathbf{V}_i \triangleq e_i + jf_i$$

$$P_i - jQ_i = (e_i - jf_i) \sum_{j=1}^{n} (G_{ij} e_j + jB_{ij} f_j) + j(G_{ij} f_j - B_{ij} e_j)$$

$$\mathbf{Y}_{ij} \triangleq G_{ij} + jB_{ij}$$

$$P_i - jQ_i = \mathbf{V}_i^* \mathbf{I}_i = \sum_{j=1}^{n} |\mathbf{V}_i||\mathbf{Y}_{ij}||\mathbf{V}_j| \angle(\theta_{ij} + \delta_i - \delta_j)$$

$$\mathbf{I}_i = \sum_{j=1}^{n} \mathbf{Y}_{ij} \mathbf{V}_{ij} \triangleq c_i + jd_i$$

$$e^{-j(\theta_{ij} + \delta_i - \delta_j)} \triangleq \cos(\theta_{ij} + \delta_i - \delta_j) - j\sin(\theta_{ij} + \delta_i - \delta_j)$$

$$P_i = \sum_{j=1}^{n} [(e_i G_{ij} e_j + jB_{ij} f_j) + f_i(G_{ij} f_j - B_{ij} e_j)]$$

$$Q_i = \sum_{j=1}^{n} [(f_i G_{ij} e_j + jB_{ij} f_j) + e_i(G_{ij} f_j - B_{ij} e_j)]$$

$$P_i = \sum_{j=1}^{n} |\mathbf{V}_i||\mathbf{Y}_{ij}||\mathbf{V}_j| \cos(\theta_{ij} + \delta_i - \delta_j)$$

$$Q_i = \sum_{j=1}^{n} |\mathbf{V}_i||\mathbf{Y}_{ij}||\mathbf{V}_j| \sin(\theta_{ij} + \delta_i - \delta_j)$$

Jacobian Matrix for Newton-Raphson Method

$$
\begin{bmatrix} \Delta P_2^{(k)} \\ \vdots \\ \Delta P_n^{(k)} \\ \hline \Delta Q_n^{(k)} \\ \vdots \\ \Delta Q_n^{(k)} \\ \hline \Delta\left|\mathbf{V}_n^{(k)}\right|^2 \end{bmatrix} =
\begin{bmatrix}
\frac{\partial P_2}{\partial e_2} & \cdots & \frac{\partial P_2}{\partial e_2} & \frac{\partial P_2}{\partial f_2} & \cdots & \frac{\partial P_2}{\partial e_n} \\
\vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial P_n}{\partial e_2} & \cdots & \frac{\partial P_n}{\partial e_n} & \frac{\partial P_2}{\partial f_2} & \cdots & \frac{\partial P_2}{\partial f_n} \\
\frac{\partial Q_2}{\partial e_2} & \cdots & \frac{\partial Q_2}{\partial e_n} & \frac{\partial Q_2}{\partial f_2} & \cdots & \frac{\partial Q_2}{\partial f_n} \\
\vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial Q_{n-1}}{\partial e_2} & \cdots & \frac{\partial Q_{n-1}}{\partial e_n} & \frac{\partial Q_{n-1}}{\partial f_2} & \cdots & \frac{\partial Q_{n-1}}{\partial f_n} \\
\frac{\partial\left|\mathbf{V}_n\right|^2}{\partial e_2} & \cdots & \frac{\partial\left|\mathbf{V}_n\right|^2}{\partial e_n} & \frac{\partial\left|\mathbf{V}_n\right|^2}{\partial f_2} & \cdots & \frac{\partial\left|\mathbf{V}_n\right|^2}{\partial f_n}
\end{bmatrix}
\begin{bmatrix} \Delta e_2^{(k)} \\ \vdots \\ \Delta e_n^{(k)} \\ \hline \Delta f_2^{(k)} \\ \vdots \\ \Delta f_{n-1}^{(k)} \\ \hline \Delta f_n^{(k)} \end{bmatrix}
$$

→

$$
\begin{bmatrix} \Delta P \\ \hline \Delta Q \\ \hline \Delta\left|\mathbf{V}\right|^2 \end{bmatrix} =
\begin{bmatrix} J_1 & J_2 \\ \hline J_3 & J_4 \\ \hline J_5 & J_6 \end{bmatrix} \times
\begin{bmatrix} \Delta e \\ \hline \Delta f \end{bmatrix}
$$

$$
\begin{bmatrix} J_1 & J_2 \\ \hline J_3 & J_4 \end{bmatrix} =
\begin{bmatrix} T & U \\ \hline U & -T \end{bmatrix}
\begin{bmatrix} c_i & d_i \\ \hline -d_i & c_i \end{bmatrix}
$$

Power Flow Formulation in Polar Coordinates

Method 1: **Four** Sub-Matrices of Jacobian

$$\begin{bmatrix} \Delta P \\ \hline \Delta Q \end{bmatrix} = \begin{bmatrix} J_1 & J_2 \\ \hline J_3 & J_4 \end{bmatrix} \begin{bmatrix} \Delta \delta \\ \hline \Delta |V| \end{bmatrix}$$

J_1

$$\frac{\partial P_i}{\partial \delta_j} = |\mathbf{V}_i||\mathbf{V}_{ij}||\mathbf{V}_j|\sin(\theta_{ij} + \delta_i - \delta_j) \quad i \neq j$$

$$\frac{\partial P_i}{\partial \delta_j} = \sum_{j=1}^{n} |\mathbf{V}_i||\mathbf{V}_{ij}||\mathbf{V}_j|\sin(\theta_{ij} + \delta_i - \delta_j)$$

$$= |\mathbf{V}_i|^2|\mathbf{Y}_{ii}|\sin\theta_{ii} - Q_i \quad i = j$$

J_2

$$\frac{\partial P_i}{\partial |\mathbf{V}_j|} = |\mathbf{V}_i||\mathbf{V}_{ij}||\mathbf{V}_j|\sin(\theta_{ij} + \delta_i - \delta_j) \quad i \neq j$$

$$\frac{\partial P_i}{\partial |\mathbf{V}_j|} = \sum_{j=1}^{n} |\mathbf{V}_j||\mathbf{V}_{ij}|\sin(\theta_{ij} + \delta_i - \delta_j) + |\mathbf{V}_i||\mathbf{Y}_{ii}|\cos\theta_{ii}$$

$$= \frac{P_i}{|\mathbf{V}_i|} + |\mathbf{V}_j||\mathbf{Y}_{ij}|\cos\theta_{ii} \quad i = j$$

J_3

$$\frac{\partial Q_i}{\partial \delta_j} = |\mathbf{V}_i||\mathbf{V}_{ij}||\mathbf{V}_j|\sin(\theta_{ij} + \delta_i - \delta_j) \quad i \neq j$$

$$\frac{\partial Q_i}{\partial \delta_j} = \sum_{j=1}^{n} |\mathbf{V}_j||\mathbf{V}_{ij}|\cos(\theta_{ij} + \delta_i - \delta_j) + |\mathbf{V}_i||\mathbf{V}_{ii}|\cos\theta_{ii}$$

$$= |\mathbf{V}_i|^2|\mathbf{Y}_{ii}|\sin\theta_{ii} + P_i \quad i = j$$

J_4

$$\frac{\partial Q_i}{\partial |\mathbf{V}_j|} = |\mathbf{V}_i||\mathbf{Y}_{ij}|\sin(\theta_{ij} + \delta_i - \delta_j) \quad i \neq j$$

$$\frac{\partial Q_i}{\partial |\mathbf{V}_j|} = |\mathbf{V}_i||\mathbf{V}_{ii}|\cos\theta_{ii} + \sum_{j=1}^{n} |\mathbf{V}_j||\mathbf{V}_{ij}|\cos(\theta_{ij} + \delta_i - \delta_j)$$

$$= |\mathbf{V}_i||\mathbf{V}_{ii}|\sin\theta_{ii} + \frac{Q_i}{|\mathbf{V}_i|} \quad i = j$$

In the event that a voltage-controlled bus is present,

4 Sub-Matrices of Jacobian

$$\begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} = \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix} \begin{bmatrix} \Delta \delta \\ \Delta |\mathbf{V}| \end{bmatrix}$$

J_5

$$|\mathbf{V}_i| = |\mathbf{V}_i|$$

$$\frac{\partial |\mathbf{V}_i|}{\partial \delta_j} = 0 \quad \text{for all} \quad i, j$$

J_6

$$\frac{\partial |\mathbf{V}_i|}{\partial |\mathbf{V}_j|} = 0 \quad i \neq j$$

$$\frac{\partial |\mathbf{V}_i|}{\partial |\mathbf{V}_i|} = 1 \quad i = j$$

$$\begin{bmatrix} \Delta P \\ \Delta Q \\ \Delta |\mathbf{V}| \end{bmatrix} = \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \\ J_5 & J_6 \end{bmatrix} \begin{bmatrix} \Delta \delta \\ \Delta |\mathbf{V}| \end{bmatrix}$$

Similarities between the elements of the submatrices of the Jacobian matrix. Therefore, let

$$\Phi_{ij} \triangleq Q_{ij} + \delta_i - \delta_j$$

$$K_{ij} \triangleq |\mathbf{V}_i||\mathbf{V}_{ij}|\cos\Phi_{ij} = |\mathbf{V}_i|G_{ij}$$

$$L_{ij} \triangleq |\mathbf{V}_i||\mathbf{V}_{ij}|\sin\Phi_{ij} = |\mathbf{V}_i|B_{ij}$$

$$K'_{ij} \triangleq |\mathbf{V}_j|K_{ij}$$

$$L'_{ij} \triangleq |\mathbf{V}_j|K_{ij}$$

$$\frac{\partial P_i}{\partial \delta_j} = L'_{ij} \quad i \neq j$$

$$\frac{\partial P_i}{\partial \delta_i} = L'_{ii} \quad i = j$$

Further simplification

J_1

$$\frac{\partial P_i}{\partial \delta_j} = L'_{ij} \quad i \neq j$$

$$\frac{\partial P_i}{\partial \delta_i} = L'_{ii} \quad i = j$$

$$[J_1] = [L'] - \begin{bmatrix} \ddots & & 0 \\ & Q_i & \\ 0 & & \ddots \end{bmatrix}$$

J_2

$$\frac{\partial P_i}{\partial |\mathbf{V}_j|} = K_{ij} \quad i \neq j$$

$$\frac{\partial P_i}{\partial |\mathbf{V}_i|} = K_{ii} + \frac{P_i}{|\mathbf{V}_i|} \quad i = j$$

$$[J_2] = [K] + \begin{bmatrix} \ddots & & 0 \\ & \frac{P_i}{|\mathbf{V}_i|} & \\ 0 & & \ddots \end{bmatrix}$$

J_3

$$\frac{\partial Q_i}{\partial \delta_j} = -K'_{ij} \quad i \neq j$$

$$\frac{\partial P_i}{\partial \delta_j} = -K'_{ij} + P_i \quad i = j$$

$$[J_3] = -[K'] + \begin{bmatrix} \ddots & & 0 \\ & P_i & \\ 0 & & \ddots \end{bmatrix}$$

J_4

$$\frac{\partial Q_i}{\partial |\mathbf{V}_j|} = L_{ij} \quad i \neq j$$

$$\frac{\partial Q_i}{\partial |\mathbf{V}_i|} = L_{ij} + \frac{Q_i}{|\mathbf{V}_i|} \quad i = j$$

$$[J_4] = [L] + \begin{bmatrix} \ddots & & 0 \\ & \frac{Q_i}{|\mathbf{V}_i|} & \\ 0 & & \ddots \end{bmatrix}$$

$$\left[\begin{array}{c|c} J_1 & J_2 \\ \hline J_3 & J_4 \end{array}\right] = \left[\begin{array}{c|c} L' & K \\ \hline -K' & L \end{array}\right] + \left[\begin{array}{c|c} -Q_i & \frac{P_i}{|\mathbf{V}_i|} \\ \hline P_i & \frac{Q_i}{|\mathbf{V}_i|} \end{array}\right]$$

$$P_i = \sum_{j=1}^{n} |\mathbf{V}_i||\mathbf{Y}_{ij}||\mathbf{V}_j|\cos(\theta_{ij} + \delta_i - \delta_j)$$

$$Q_i = \sum_{j=1}^{n} |\mathbf{V}_i||\mathbf{Y}_{ij}||\mathbf{V}_j|\sin(\theta_{ij} + \delta_i - \delta_j)$$

Power Flow Formulation in Polar Coordinates

Method 2: **Six** Sub-Matrices of Jacobian

$$[J_2] = [K] + \begin{bmatrix} \ddots & & 0 \\ & \frac{P_i}{|\mathbf{V}_i|} & \\ 0 & & \ddots \end{bmatrix}$$

J_2

$$[J_2] = [K'] + \begin{bmatrix} \ddots & & 0 \\ & P_i & \\ 0 & & \ddots \end{bmatrix}$$

J_1

$$\frac{\partial P_i}{\partial \delta_j} = L'_{ij} \quad i \neq j$$

$$\frac{\partial P_i}{\partial \delta_i} = L'_{ii} \quad i = j$$

$$[J_1] = [L'] - \begin{bmatrix} \ddots & & 0 \\ & Q_i & \\ 0 & & \ddots \end{bmatrix}$$

$$[J_4] = [L] + \begin{bmatrix} \ddots & & 0 \\ & \frac{Q_i}{|\mathbf{V}_i|} & \\ 0 & & \ddots \end{bmatrix}$$

J_3

$$\frac{\partial Q_i}{\partial \delta_j} = -K'_{ij} \quad i \neq j$$

$$\frac{\partial P_i}{\partial \delta_j} = -K'_{ij} + P_i \quad i = j$$

$$[J_3] = -[K'] + \begin{bmatrix} \ddots & & 0 \\ & P_i & \\ 0 & & \ddots \end{bmatrix}$$

J_4

$$[J_4] = [L'] + \begin{bmatrix} \ddots & & 0 \\ & Q_i & \\ 0 & & \ddots \end{bmatrix}$$

$$\left[\begin{array}{c|c} J_1 & J_2 \\ \hline J_3 & J_4 \end{array}\right] = \left[\begin{array}{c|c} L' & K \\ \hline -K' & L \end{array}\right] + \left[\begin{array}{c|c} -Q_i & \frac{P_i}{|\mathbf{V}_i|} \\ \hline P_i & \frac{Q_i}{|\mathbf{V}_i|} \end{array}\right]$$

$$\left[\begin{array}{c|c} J_1 & J_2 \\ \hline J_3 & J_4 \\ \hline J_5 & J_6 \end{array}\right] = \left[\begin{array}{c|c} L' & K' \\ \hline -K' & L' \\ \hline 0 & |\mathbf{V}_i| \end{array}\right] + \left[\begin{array}{c|c} -Q_i & P_i \\ \hline P_i & Q_i \\ \hline 0 & 0 \end{array}\right]$$

Fast Decoupled Method

$$\left[\frac{\Delta P}{\Delta Q}\right] = \left[\begin{array}{c|c} J_1 & J_2 \\ \hline J_3 & J_4 \end{array}\right] \left[\frac{\Delta\delta}{\frac{\Delta|\mathbf{V}|}{|\mathbf{V}|}}\right]$$

Conforms with typical representation of notations

$$\left[\frac{\Delta P}{\Delta Q}\right] = \left[\begin{array}{c|c} H & N \\ \hline J & L \end{array}\right] \left[\frac{\Delta\delta}{\frac{\Delta|\mathbf{V}|}{|\mathbf{V}|}}\right]$$

Off diagonal elements set to 0s.

$$\left[\frac{\Delta P}{\Delta Q}\right] = \left[\begin{array}{c|c} H & 0 \\ \hline 0 & L \end{array}\right] \left[\frac{\Delta\delta}{\frac{\Delta|\mathbf{V}|}{|\mathbf{V}|}}\right]$$

Fast Decoupled Method

$$\begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} = \begin{bmatrix} H & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \dfrac{\Delta|\mathbf{V}|}{|\mathbf{V}|} \end{bmatrix}$$

Expand

$$[\Delta P] = [H][\Delta\delta]$$

$$[\Delta Q] = [L]\left[\frac{\Delta|V|}{|V|}\right]$$

Inverse

$$[\Delta\delta] = [H]^{-1}[\Delta P]$$

$$\left[\frac{\Delta|V|}{|V|}\right] = [L]^{-1}[\Delta Q]$$

H

$$H_{ij} = V_iV_j(G_{ij}\sin\delta_{ij} - B_{ij}\cos\delta_{ij}) \quad i \neq j$$

$$H_{ii} = -B_{ii}V_i^2 - Q_i \quad i = j$$

L

$$L_{ij} = V_iV_j(G_{ij}\sin\delta_{ij} - B_{ij}\cos\delta_{ij}) = H_{ij} \quad i \neq j$$

$$L_{ii} = -B_{ii}V_i^2 - Q_i \quad i = j$$

$$G_{ij}\sin\delta_{ij} \ll B_{ij} \qquad \sin\delta_{ij} = \sin(\delta_i - \delta_j) \cong \delta_i - \delta_j = \delta_{ij}$$

$$Q_i \ll B_{ii}V_i^2 \qquad \cos\delta_{ij} = \cos(\delta_i - \delta_j) \cong 1.0$$

Integrating simplified terms

$$[\Delta P] = [V \times B' \times V][\Delta\delta]$$

$$[\Delta Q] = [V \times B'' \times V]\left[\frac{\Delta\delta}{V}\right]$$

Define susceptance

$$B'_{ij} = -\frac{1}{X_{ij}} \quad i \neq j$$

$$B'_{ii} = \sum_{j=1}^{n}\frac{1}{X_{ij}} \quad i \neq j$$

$$B''_{ij} = -B_{ij}$$

Fast Decoupled Method

$$[\Delta P] = [V \times B' \times V][\Delta \delta]$$

$$[\Delta Q] = [V \times B'' \times V]\left[\frac{\Delta \delta}{V}\right]$$

$$\rightarrow$$

$$\left[\frac{\Delta P}{V}\right] = [B'][\Delta \delta]$$

$$\left[\frac{\Delta Q}{V}\right] = [B''][\Delta V]$$

- Both susceptions are real, sparse, and symmetrical
- Shunt susceptance, transformer off-nominal taps, and phase shifts are omitted on these equations
- Constant sparse upper triangular factors can be computed and stored only once at the start of the solution

"DC" Power Flow

$$P_i = \sum_{j=1}^{n} |\mathbf{V}_i||\mathbf{Y}_{ij}||\mathbf{V}_j|\cos(\theta_{ij} + \delta_i - \delta_j)$$

$$Q_i = \sum_{j=1}^{n} |\mathbf{V}_i||\mathbf{Y}_{ij}||\mathbf{V}_j|\sin(\theta_{ij} + \delta_i - \delta_j)$$

$$P_{ij} = \frac{V_i V_j}{Z_{ij}} \sin(\delta - \delta_j)$$

Where $\mathbf{v}_i = |v_i|\angle\delta_i \quad \mathbf{v}_j = |v_j|\angle\delta_j$

- Linearized power flow from the non-linear system

$X_{ij} \cong \mathbf{Z}_{ij}$ since $X_{ij} \gg R_{ij}$

$|v_i| \cong 1.0$ pu

$|v_j| \cong 1.0$ pu

$\sin(\delta_i - \delta_j) \cong \delta_i - \delta_j$

$$P_{ij} \cong \frac{\delta_i - \delta_j}{X_{ij}} \cong B_{ij}(\delta_i - \delta_j)$$

Simplified

$$[P] = [B][\delta]$$

Inversely…

$$[\delta] = [B]^{-1}[P]$$