Exploring multiple solutions to power flow and optimal power flow problems

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Comments on Optimal Power Flow

At a high level, the optimal power flow problem allocates and schedules dispatchable resources in the power grid to achieve some objective subject to some constraints.

There are many, many variants…

In this talk we will only consider the relatively “easy” classical OPF.

Classical OPF Problem

$$\min \sum_{k \in \mathcal{G}} f_k\left(P_{Gk}\right) = \sum_{k \in \mathcal{G}} c_{2k} P_{Gk}^2 + c_{1k} P_{Gk} + c_{0k}$$

Cost

subject to

$$P_{Gk}^{\min} \leq P_{Gk} \leq P_{Gk}^{\max}$$

$$Q_{Gk}^{\min} \leq Q_{Gk} \leq Q_{Gk}^{\max}$$

$$\left(V_k^{\min}\right)^2 \leq V_{dk}^2 + V_{qk}^2 \leq \left(V_k^{\max}\right)^2$$

$$\left|S_{lm}\right| \leq S_{lm}^{\max}$$

Engineering Constraints

$$P_{Gk} - P_{Dk} = V_{dk} \sum_{i=1}^{n} \left(G_{ik} V_{di} - B_{ik} V_{qi}\right) + V_{qk} \sum_{i=1}^{n} \left(B_{ik} V_{di} + G_{ik} V_{qi}\right)$$

$$Q_{Gk} - Q_{Dk} = V_{dk} \sum_{i=1}^{n} \left(-B_{ik} V_{di} - G_{ik} V_{qi}\right) + V_{qk} \sum_{i=1}^{n} \left(G_{ik} V_{di} - B_{ik} V_{qi}\right)$$

Physical Laws

Motivation for a new approach

This problem is not “easy.”

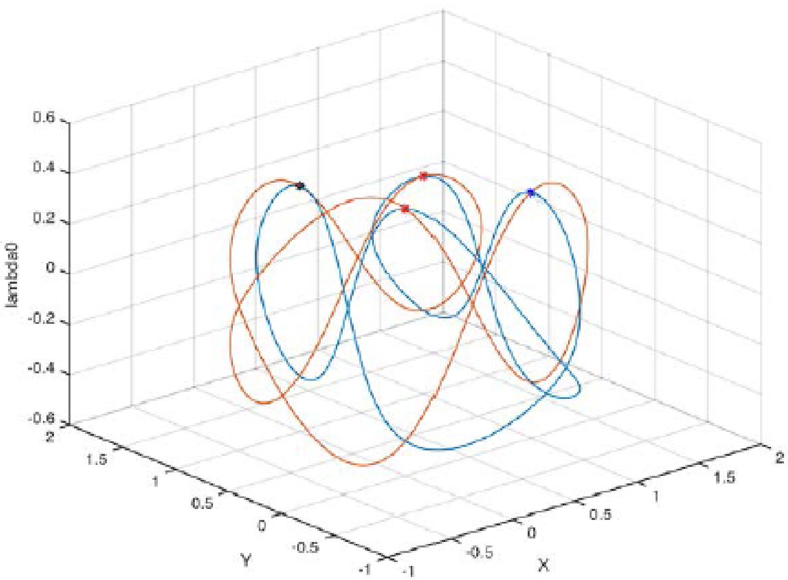
Recent promising work includes a convex relaxation of the problem and SDP to find a global solution… sometimes.

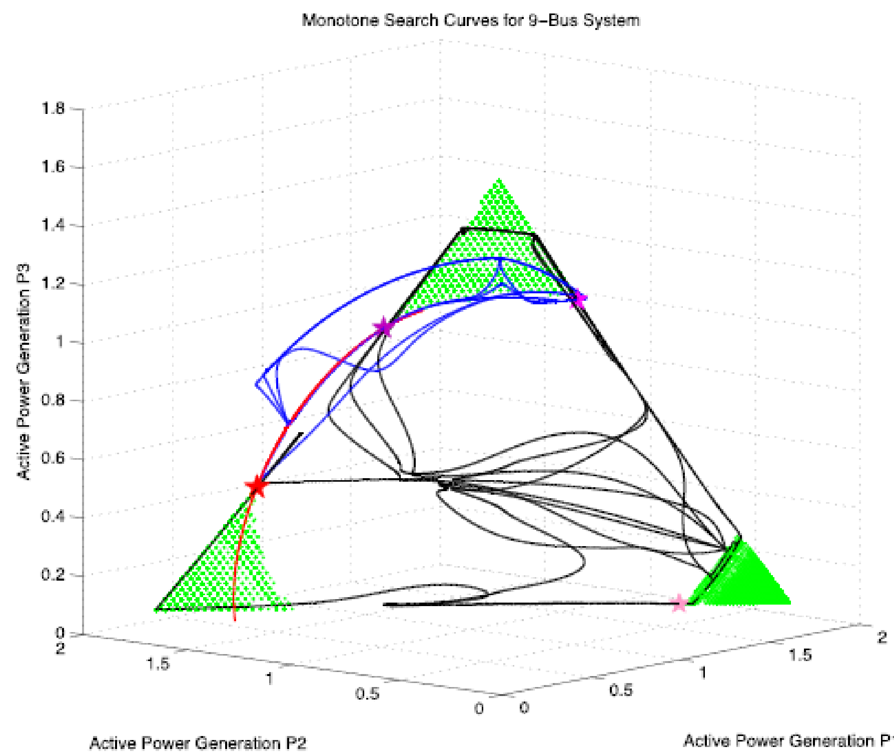
How might we seek out better solutions (if any) when a convex relaxation doesn’t yield a global solution?

How might we find other solutions to compare against an optimal solution if a convex relaxation does yield a global solution?

Our Approach: Enumeration

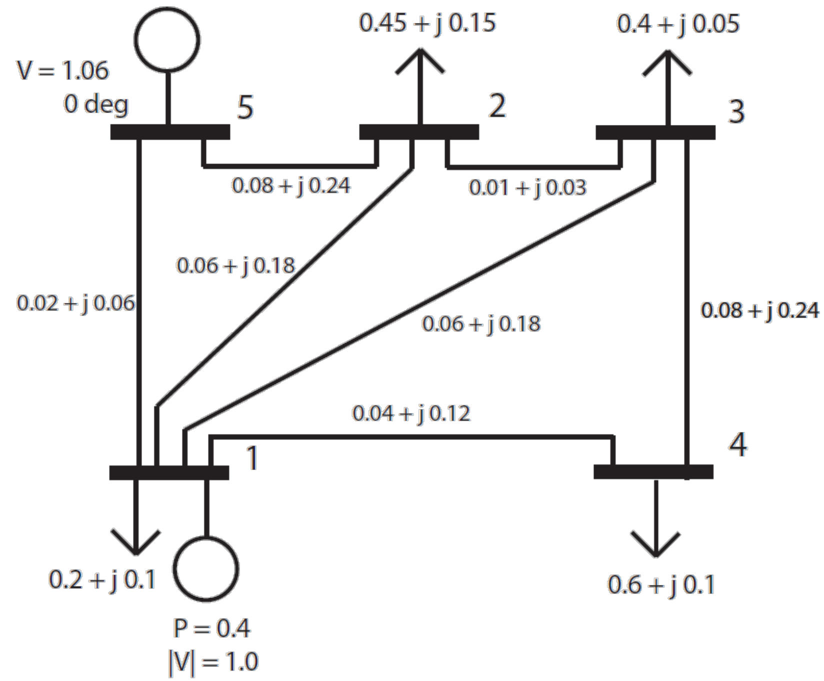
We develop a method to trace between critical points of the optimization problem.





The Power Flow Problem

To introduce the tracing method, we revisit the standard power flow problem and seek to find (all) multiple solutions.



$$P_i = \sum_{k=1}^{N} V_i V_k \left[G_{ik} cos(\theta_i - \theta_k) + B_{ik} sin((\theta_i - \theta_k)\right]$$

$$Q_i = \sum_{k=1}^{N} V_i V_k \left[G_{ik} sin(\theta_i - \theta_k) - B_{ik} cos((\theta_i - \theta_k)\right]$$

(Polar Coordinates)

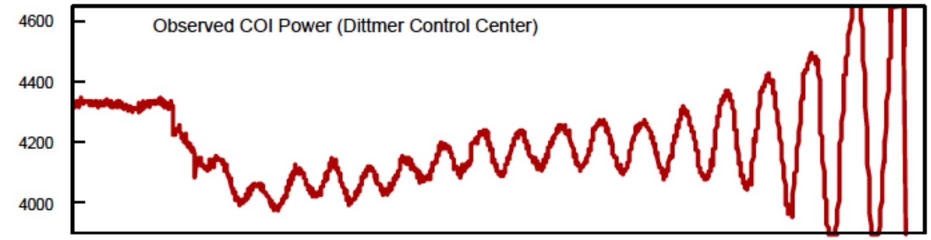
The Power Flow Problem

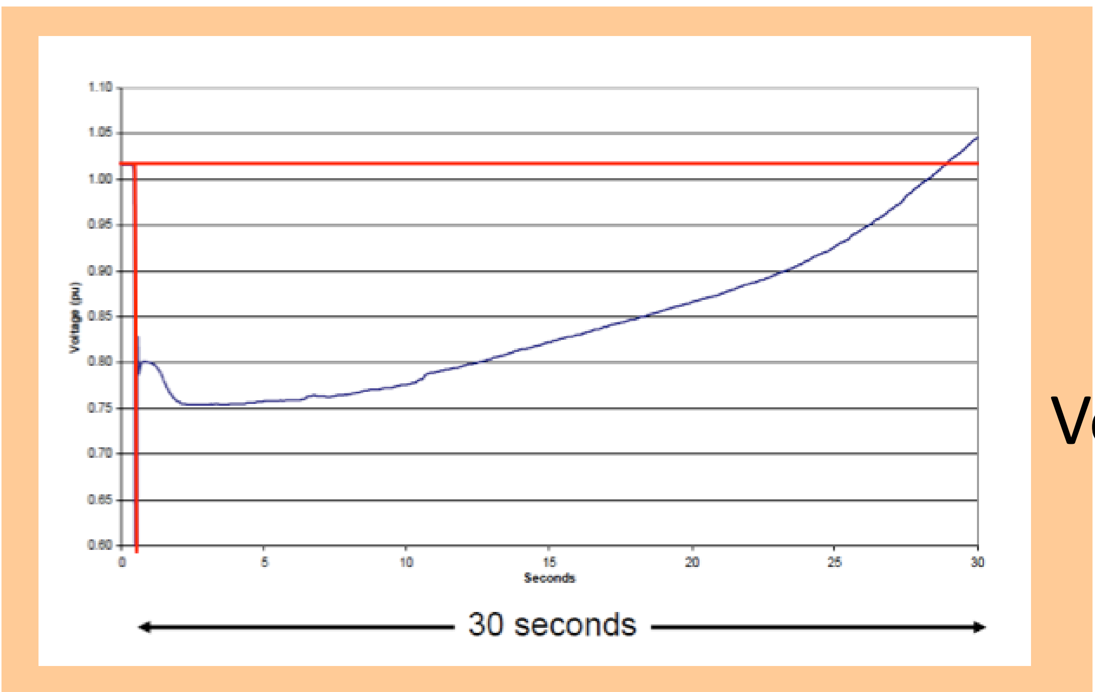
The power flow problem is interesting and important on its own:

- Used for real-time studies. What happens if ...
- Solutions are equilbria for a dynamic model.
- Multiple equilibria aid in direct angle stability and voltage stability analyses.
- Specialized studies

Life at the Physical Layer

"Angle" stability problems

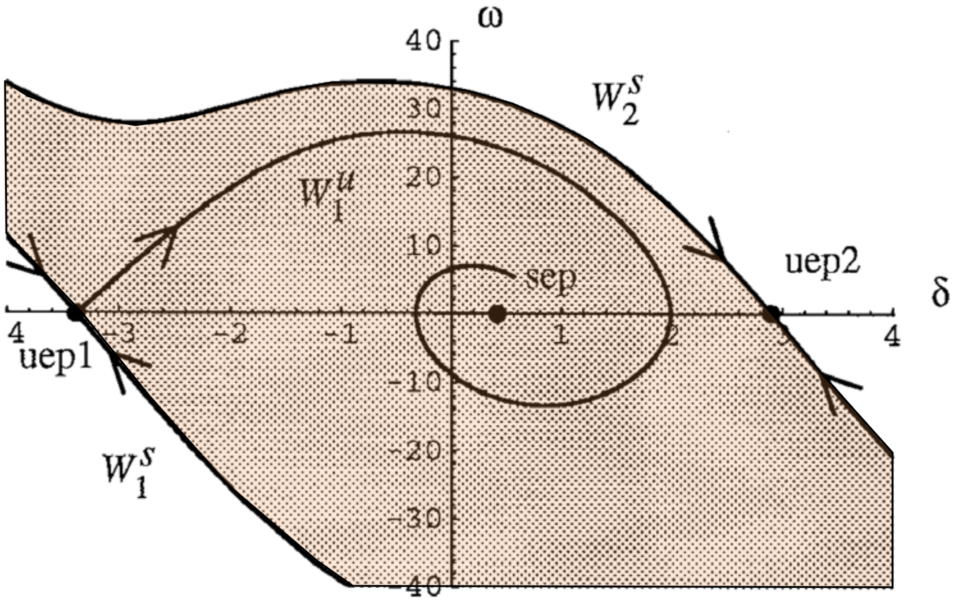


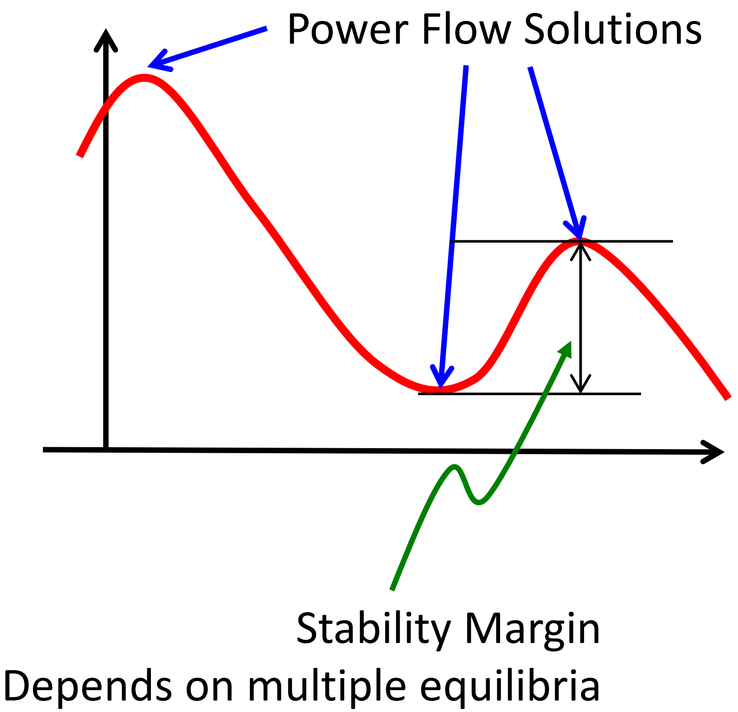


Voltage Stability Problems

Power Flow Solutions are Dynamic Equilibria

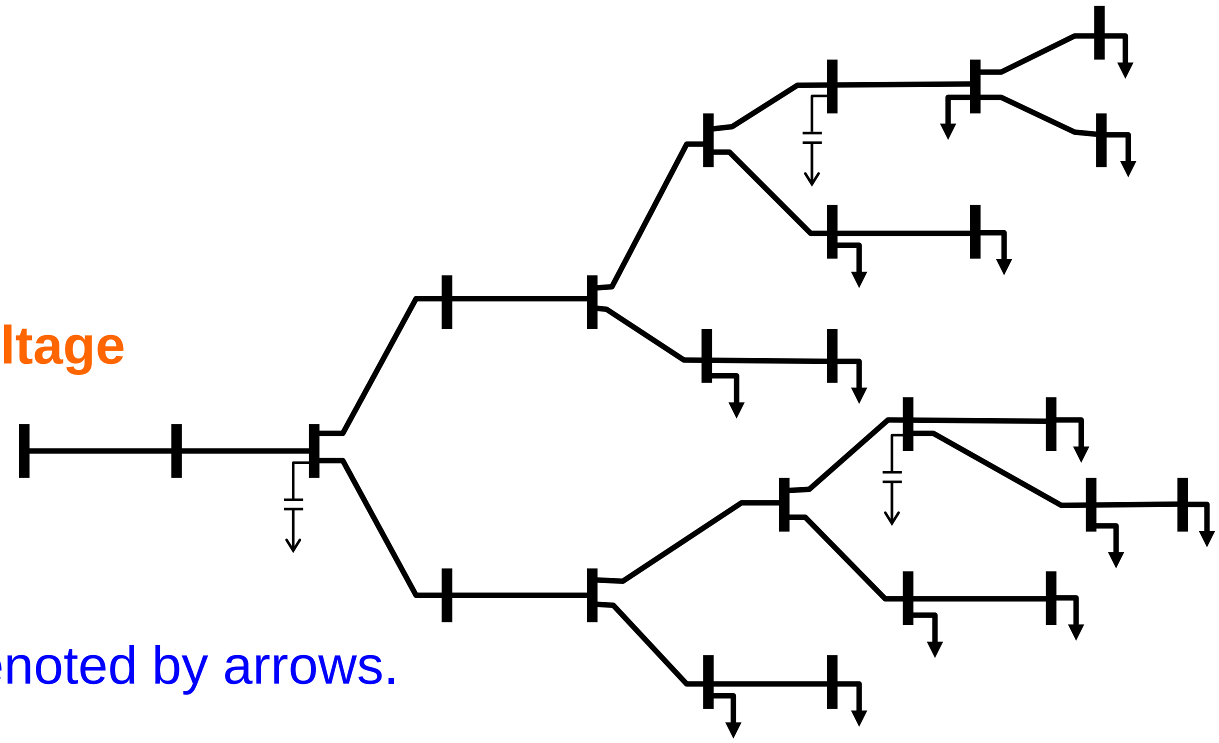
The Power Flow solutions are the dynamic equilibriums: Needed for energy function analysis.





The boundary of the region of attraction is the union of stable manifolds of certain type-1 unstable equilibrium points.

FIDVR Cascade: Do All Motors Stall?

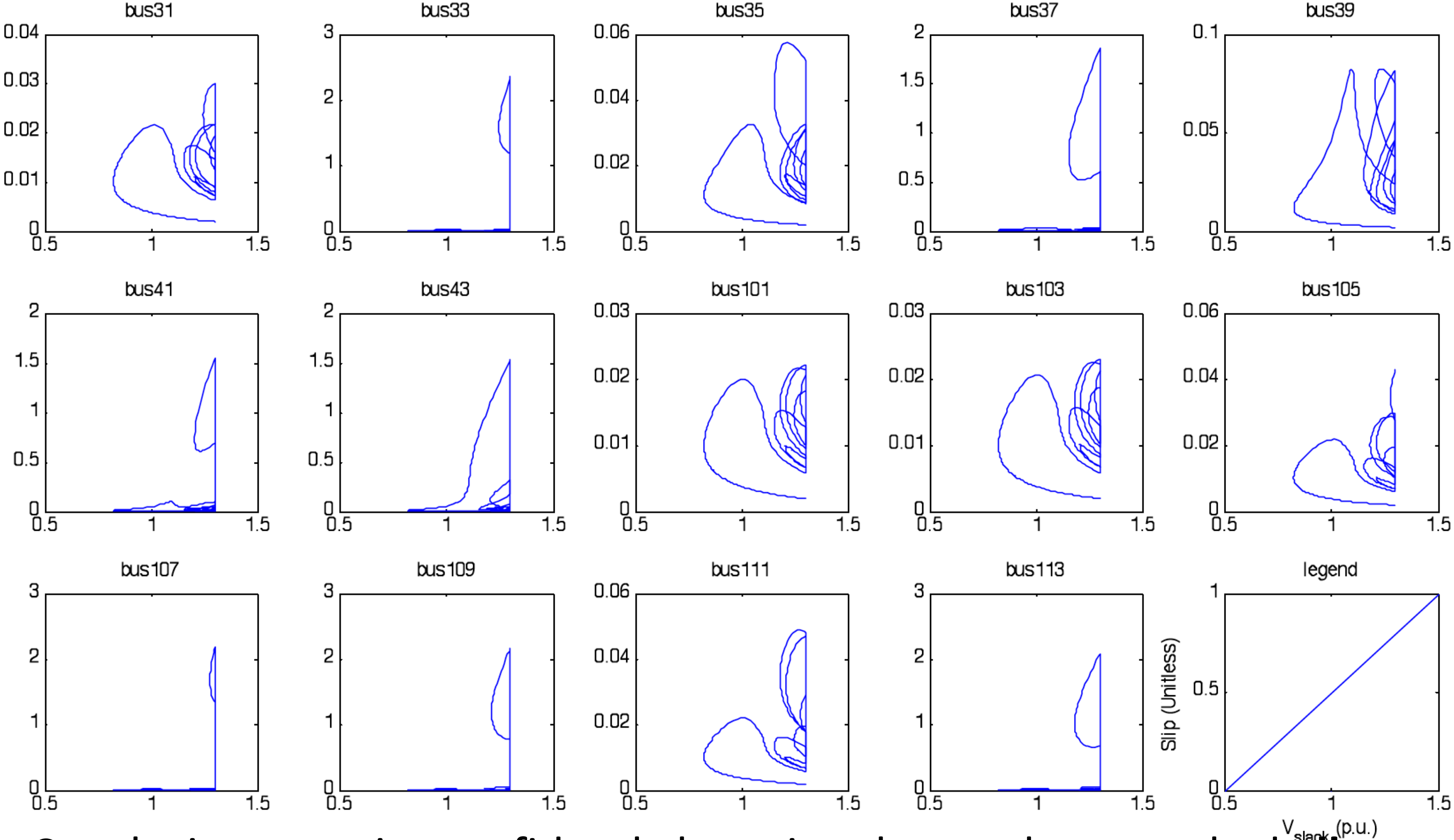


Loads denoted by arrows.

25 Buses, 13 loads, tree distribution network, single connection to the grid.

Is it possible for a fraction of motors to stall in this network without stalling them all?

Bifurcation Diagrams: slip vs. voltage



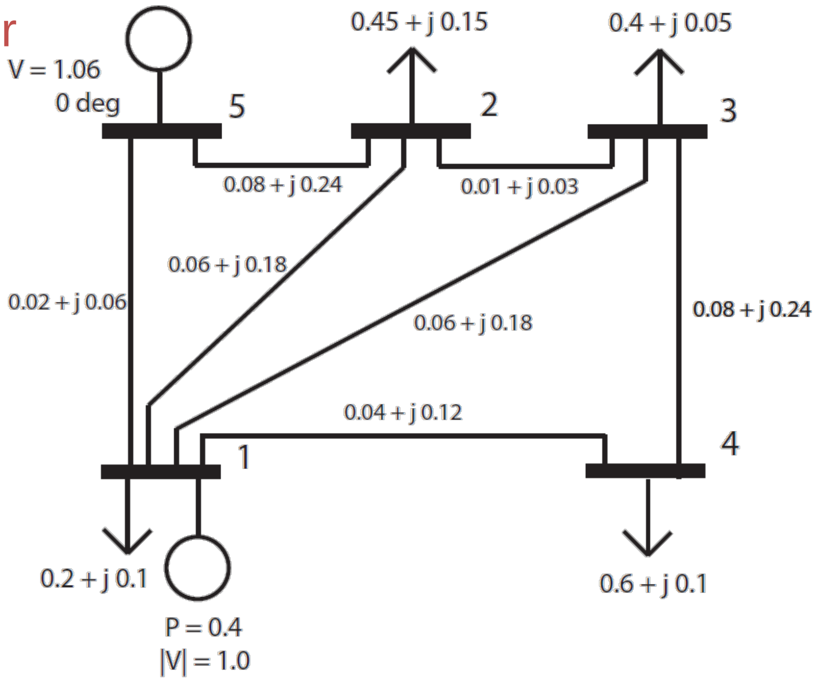
Conclusions require confidently knowing that we have tracked **all** possible solutions

Traditional Power Flow Problem

Textbook approach

Specify

- Load Active Power and Load Reactive Power
- Generator Voltage Magnitude and Active Power
- One “slack bus” with Voltage Magnitude and Angle



Solve for

- Load Voltage Magnitude and Angle
- Generator Reactive Power and Angle
- Slack bus Active and Reactive Power

Observation:

Tight constraints on

- load bus powers, gen bus active power and voltage, and slack bus voltage and angle.

Loose constraints on

- load bus voltages, gen bus reactive power, slack bus active and reactive power, and line flows.

Finding All the Solutions

Solve the Power Flow Equations… completely.

$$P_i = \sum_{k=1}^{N} V_i V_k \left[G_{ik} cos(\theta_i - \theta_k) + B_{ik} sin((\theta_i - \theta_k) \right]$$

$$Q_i = \sum_{k=1}^{N} V_i V_k \left[G_{ik} sin(\theta_i - \theta_k) - B_{ik} cos((\theta_i - \theta_k) \right]$$

This has been studied for a long time.

It turns out to be a hard problem…

Previous Work

18th Century **Bezout (1779):** Applied to Power Systems $2^{2(N-1)}$

$$P_{Gk} - P_{Dk} = V_{dk} \sum_{i=1}^{n} (G_{ik} V_{di} - B_{ik} V_{qi}) + V_{qk} \sum_{i=1}^{n} (B_{ik} V_{di} - G_{ik} V_{qi})$$

$$Q_{Gk} - Q_{Dk} = V_{dk} \sum_{i=1}^{n} (-B_{ik} V_{di} - G_{ik} V_{qi}) + V_{qk} \sum_{i=1}^{n} (G_{ik} V_{di} - B_{ik} V_{qi})$$

In rectangular coordinates the power flow equations form coupled quadratic equations. The number of possible complex solutions are given by Bezout:

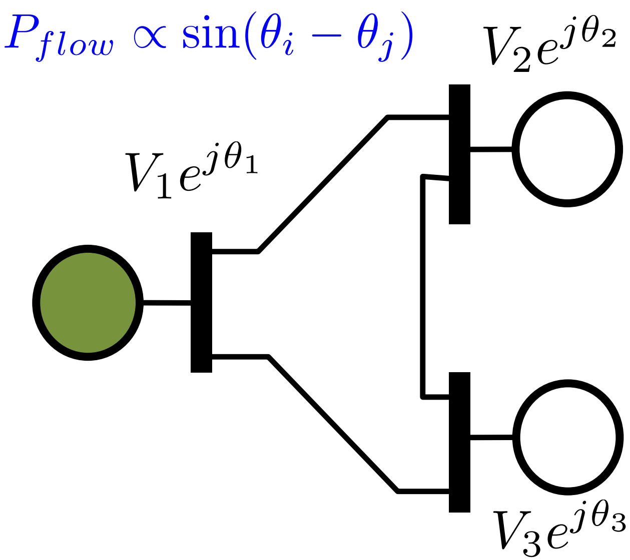
$$2^{2(N-1)}$$

That's a lot of solutions!

Previous Work

20th Century

Tavora & Smith (1972): 3-bus example admits 0, 2, 4 & 6 solutions.



For lossless system, unloaded conditions, there are 6 solutions

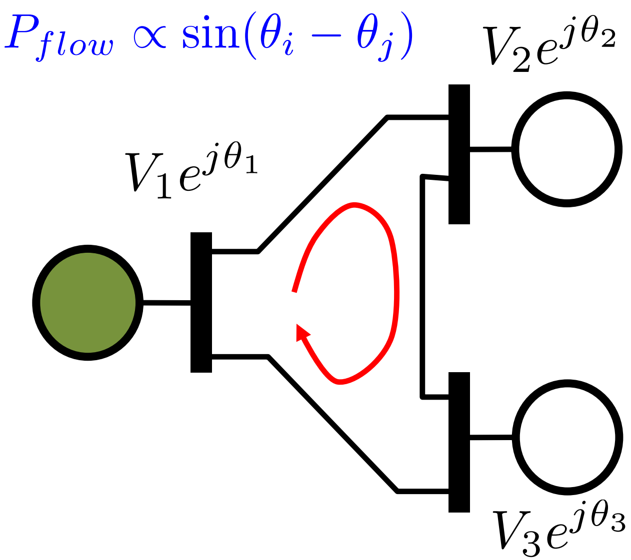
Note: All three buses are "PV" buses. Voltage Magnitudes are fixed, voltage angle are unknown except for reference bus.

θ_1	θ_2	θ_3
0°	0°	0°
0°	180°	0°
0°	0°	180°
0°	180°	180°
0°	120°	240°
0°	240°	120°

Previous Work

20th Century

Tavora & Smith (1972): 3-bus example admits 0, 2, 4 & 6 solutions.



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θ_1	θ_2	θ_3
$0°$	$0°$	$0°$
$0°$	$180°$	$0°$
$0°$	$0°$	$180°$
$0°$	$180°$	$180°$
$0°$	$120°$	$240°$
$0°$	$240°$	$120°$

Previous Work

20th Century

Baillieul & Byrnes (1982): bound the number of solutions by

$$\binom{2(N-1)}{N-1}$$

Still a large number.

This is consistent with number of solutions found by Tavora and Smith.

However… these are the number of complex solutions. Empirically for N>3, the number of real-valued solutions appears to be lower.

Knowledge of a bound, doesn't find the solutions.

Open question: does there exist a valid power system model with the number of real-valued solutions equal to the B&B bound?

Previous Work

20th Century

Salam (1989): Globally convergent probability–one homotopy method:

$$f(x) = \lambda f_0(x) + (1 - \lambda) f_1(x)$$

Trace solutions as a function λ.

A trace is required for each complex solution. Sift for real-valued solutions.

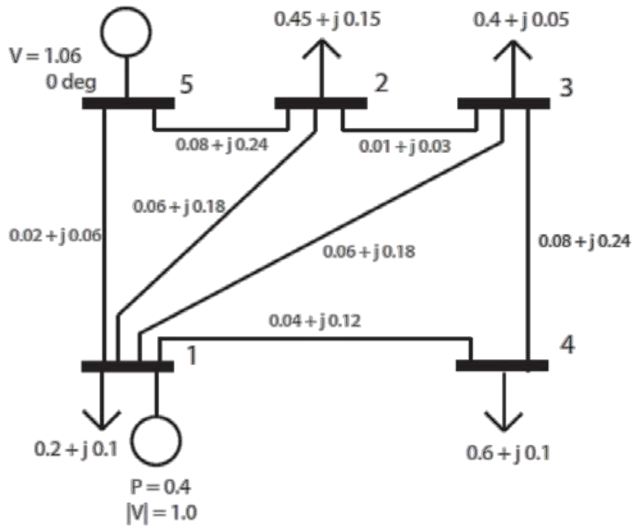
(This must be done carefully, but it does work.)

Method scales with the number of complex solutions.

Previous Work

20[th] Century

Salam (1989): Examples:



5 Bus system: 70 traces, 10 real-valued solutions.

7 Bus system: 924 traces, 4 real-valued solutions.

Empirically, the number of real-valued solutions appears to be much smaller than the bound.

Previous Work

20th Century

Tracing Methods

Hiskens (1990) Trace between solutions by varying parameters in the power flow equations.

Ma & Thorp (1991,1993): **Efficient method**, S*N continuation traces to find ***all*** the real-valued solutions.

Method scales with the number of real-valued solutions.

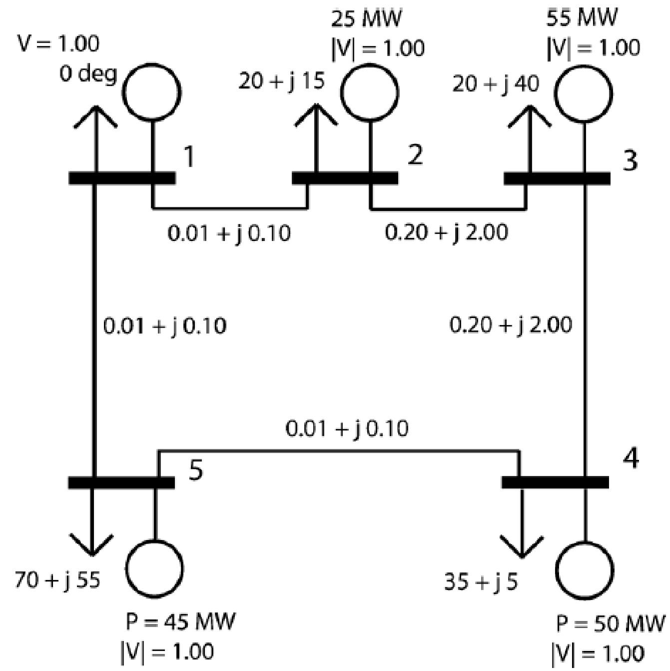
Some Previous Work

21th Century

Molzahn & Lesieutre (2013): 5-bus **counter example** to Ma-Thorp completeness claim.

TABLE I
ALL SOLUTIONS TO THE FIVE-BUS SYSTEM

Solution	δ_1	δ_2	δ_3	δ_4	δ_5
1	0	1.286	22.061	2.194	0.372
2	0	0.166	171.198	0.028	-0.710
3	0	-169.906	-148.192	-167.129	-168.909
4	0	-168.702	3.182	-167.131	-167.912
5	0	2.187	45.923	46.616	-143.973
6	0	-168.657	-172.863	44.012	-145.341
7	0	-171.391	-99.227	50.716	-141.807
8	0	-0.897	-168.405	44.388	-145.144
9	0	-169.370	-10.988	165.903	-25.378
10	0	-169.282	-160.897	166.147	-22.898



Some Previous Work

21th Century

- **Molzahn & Lesieutre (2013)**: 5-bus counter example to Ma-Thorp method
- **Mehta, Nguyen & Turitsyn (2014)**: trace 49 Million homotopies on IEEE 14-bus to find 30 real solutions
- **Molzahn, Mehta, & Niemerg (2016):** topological influence on number of solutions

Lesieutre and Wu (2015) locate all the 30 solutions of IEEE 14-bus system with 299 traces

Some Previous Work

21th Century

- **Coss, Hauenstein, Hong, Molzahn (2017)**:
  Real solutions to a rank 1 Kuramoto model
- **Chen, Davis, Mehta (2017):** Bounds for tree and ring networks: respectively

$$2^{N-1}$$

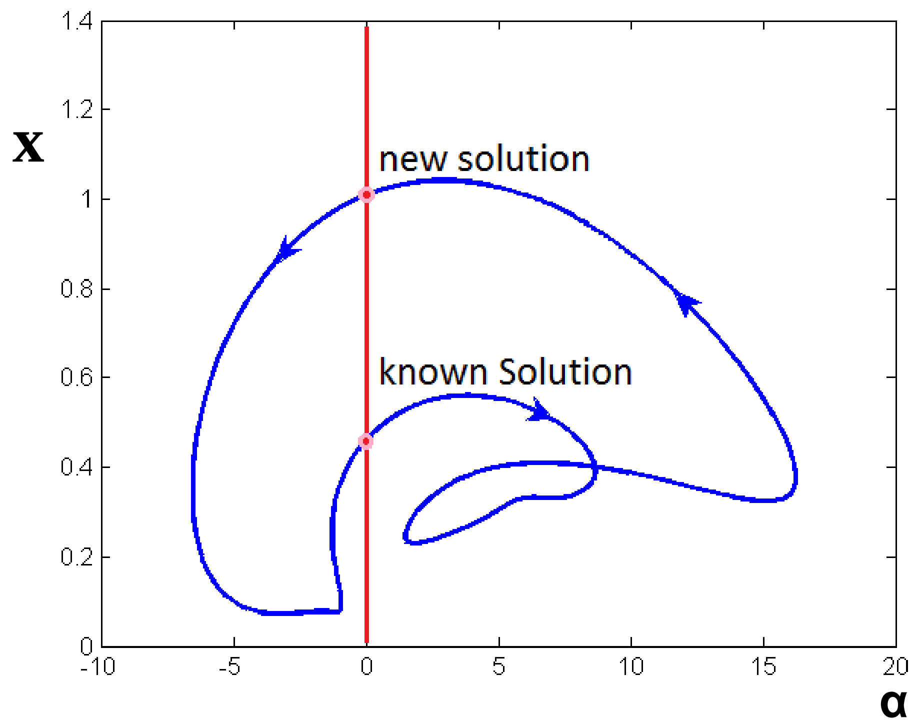
$$N\binom{N-1}{(N-1)/2}$$

Wu, Molzahn, Lesieutre, Dvijotham (2017)
cast AC OPF as intersecting ellipsoids

Revisit the Tracing Method

Form traces using power flow active power equations (and possibly reactive power equations)

$$
\begin{aligned}
f_1(x) &= 0\\
&\vdots\\
f_i(x) &= 0 + \alpha\\
&\vdots\\
f_n(x) &= 0
\end{aligned}
$$

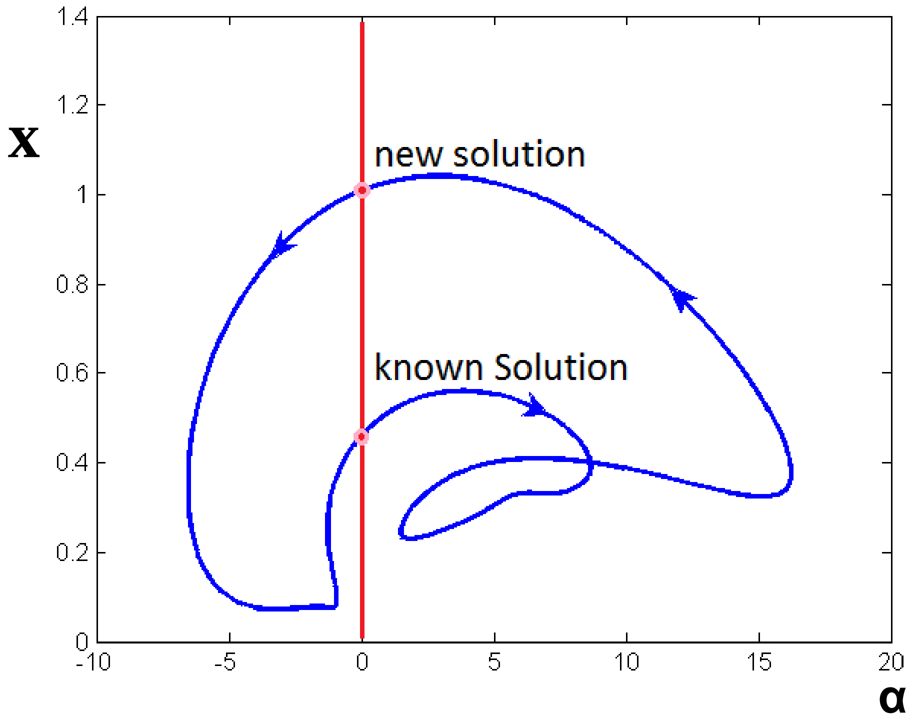


- continuously change α
- Every Point at which $\alpha = 0$ is a solution to the power flow equations.
- They prove that these particular traces are bounded and must form loops.
- They prove that all the solutions are connected by these loops.

Revisit the Tracing Method

Form traces using power flow active power equations (and possibly reactive power equations)

$$
\begin{aligned}
f_1(x) &= 0 \\
&\vdots \\
f_i(x) &= 0 + \alpha \\
&\vdots \\
f_n(x) &= 0
\end{aligned}
$$



This is spectacularly efficient!

Revisit the Tracing Method

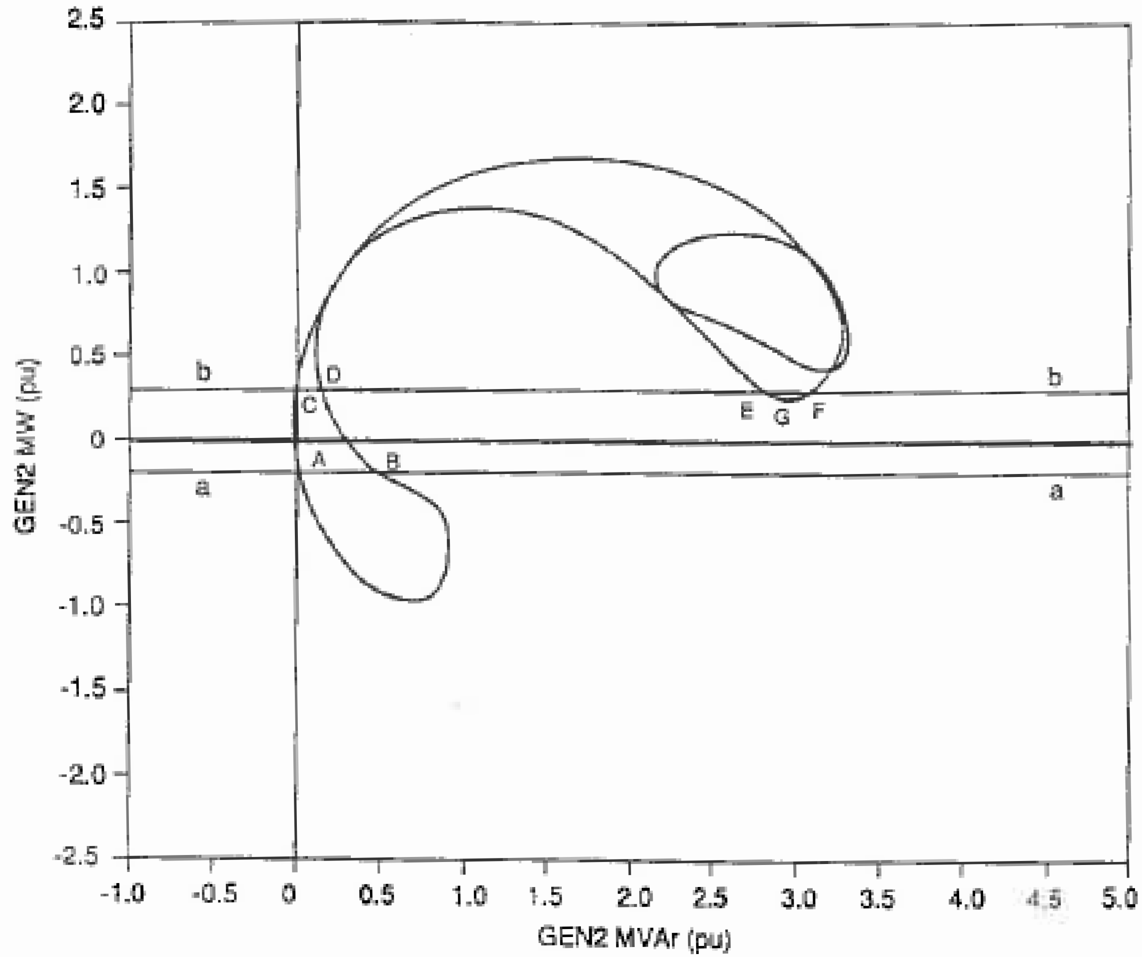


Figure 7.1: Generator Power 1-Manifold.

I. Hisken, "Energy Functions, Transient Stability and Voltage Behaviour In Power Systems," Ph.D Dissertation, 1990, p 179.

Revisit the Tracing Method

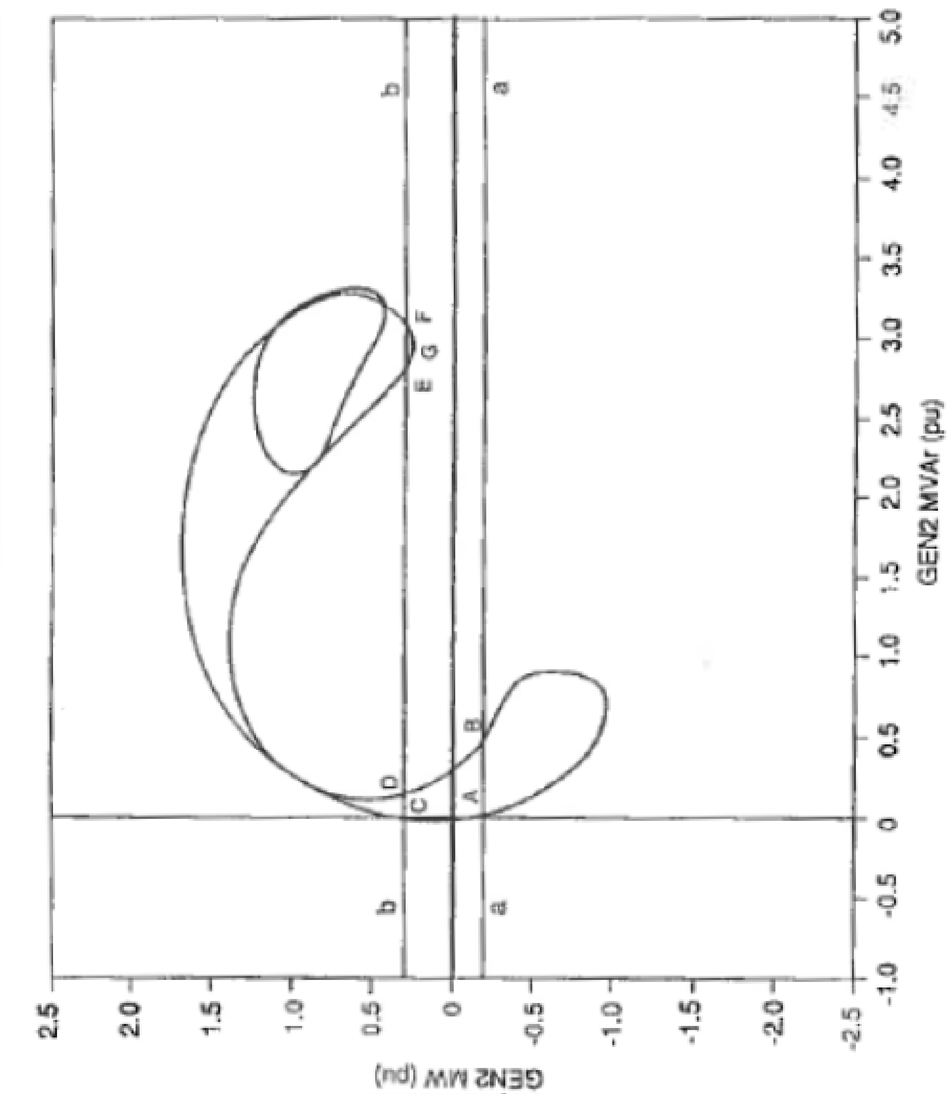


Figure 7.1: Generator Power 1-Manifold.

I. Hisken, “Energy Functions, Transient Stability and Voltage Behaviour In Power Systems,” Ph.D Dissertation, 1990, p 179.

Extend the Tracing Method

Increase the number of paths to be traveled:

- **Include Voltage Magnitude Equations**

  (more equations)

- **Represent the Power Flow Equations at Ellipses**
  (bounded, closed surface)

- **Repeat tracing method**

Main Result: the Power Flow Equations can be expressed as high dimensional ellipses.

The Power Flow Equations in Rectangular Coordinates

$$\mathbf{V}^{\mathrm{T}}\mathbf{M}_{\mathrm{pi}}\mathbf{V} = \mathrm{P}_{\mathrm{i}},$$
$$\mathbf{V}^{\mathrm{T}}\mathbf{M}_{\mathrm{qj}}\mathbf{V} = \mathrm{Q}_{\mathrm{j}},$$
$$\mathbf{V}^{\mathrm{T}}\mathbf{M}_{\mathrm{vk}}\mathbf{V} = \mathrm{V}_{\mathrm{mk}}^{2}$$

where $\mathbf{V} = [\mathbf{V}_{\mathrm{q}}^{\mathrm{T}}\ \mathbf{V}_{\mathrm{d}}^{\mathrm{T}}]^{\mathrm{T}}$; *i* is the index for all the buses excluding the slack bus; *j* is the index for PQ bus; *k* is the index for PV bus.

$\mathbf{M}_{\mathrm{pi}}, \mathbf{M}_{\mathrm{qj}}$ **are symmetric rank-4 matrices with two repeated positive eigenvalues and two repeated negative eigenvalues.** [James Foster'13] **NOT ellipses.**

Ellipses!

Linear Combinations of these equations are Ellipses. In particular,

$$T_o = \gamma \sum_k M_{vk} + \sum_j M_{qj} \succ 0$$

Define the positive definite matrix T for a base ellipse:

$$T = \beta T_o + \sum_m M_m / p_m \succ 0$$

and form power system equations as perturbations of a base ellipse

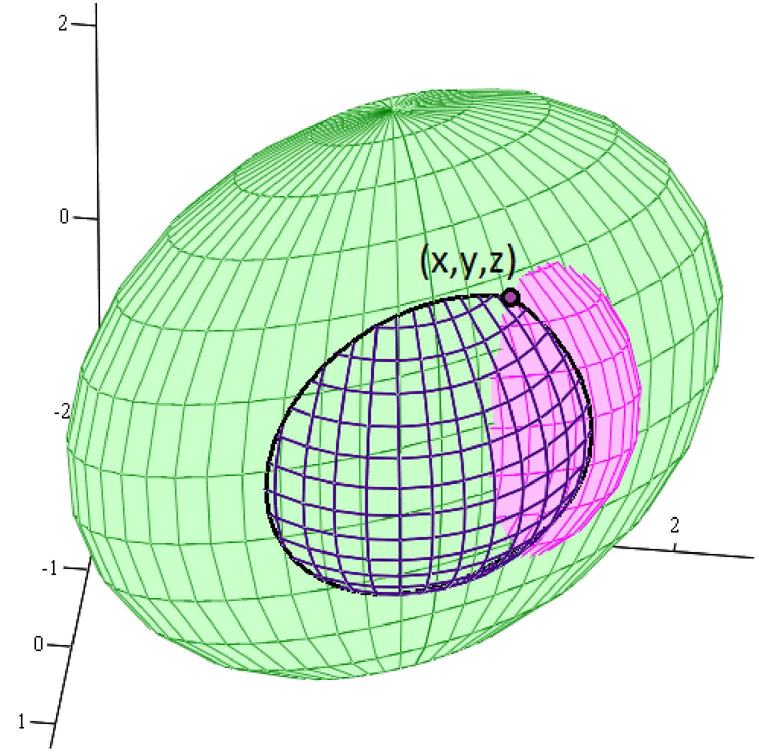
Small Print: may not work for all systems. When in doubt, set up SDP:

$$T_0 = \sum_k \alpha_k M_{vk} + \sum_j \beta_j M_{qj} + \sum_i \gamma_i M_{pi} \succ 0$$

Power Flow Ellipses

$$V^T\left(T+\epsilon M_{pi}\right)V=b_{po}$$
$$V^T\left(T+\epsilon M_{qj}\right)V=b_{qj}$$
$$V^T\left(T+\epsilon M_{vk}\right)V=b_{vk}$$

Now apply the tracing algorithm



Results

Apply the High-Dimensional Elliptical structure to systems whose solution sets are known.

Cases	# real solutions	# paths traced	Baillieul Bound	Bezout Bound
Tavora & Smith 3 bus	6	**8**	6	16
Baillieul & Byrnes 4 bus	12	**18**	20	64
Wisconsin 5 bus	10	**38**	70	256
Salam 5 bus	10	**28**	70	256
Salam 7 bus	4	**21**	924	4096
IEEE 14 bus	30	**299**	10400600	67108864

Results

New benchmark solutions!

(currently unverifiable)

Cases	# real solutions	# paths traced	Baillieul Bound	Bezout Bound
IEEE 30 Bus	472	9780	$5.9*10^{16}$	$2.9*10^{17}$
IEEE 39 Bus	176	5226	$6.9*10^{21}$	$7.6*10^{22}$

Application to the OPF problem

Can we represent the solution to the OPF problem as ellipses somehow?

How about the first-order conditions? Then the tracing routine may locate new solution.

Consider

$$\begin{aligned} &\min_x f(x) \\ \text{s.t. } &h(x) = 0 \\ &g(x) \leq 0 \quad \rightarrow \quad g(x) + s^2 = 0 \end{aligned}$$

First-Order Conditions

$$L = f(x) + \lambda_1 h(x) + \lambda_2 \left(g(x) + s^2\right)$$

Yields first-order conditions

$$\begin{aligned} \frac{\partial f(x)}{\partial x} + \lambda_1 \frac{\partial h(x)}{\partial x} + \lambda_2 \frac{\partial g(x)}{\partial x} &= 0 \\ h(x) &= 0 \\ g(x) + s^2 &= 0 \\ 2s\lambda_2 &= 0 \end{aligned}$$

Unfortunately, there is no way to combine these equations to get a base ellipse.

The trick

Scale the cost

$$\min_x \lambda_0 f(x)$$
$$\text{s.t. } h(x) = 0$$
$$g(x) + s^2 = 0$$

Scaling should be positive.

Yields the (Fritz John) first-order conditions

$$\lambda_0 \frac{\partial f(x)}{\partial x} + \lambda_1 \frac{\partial h(x)}{\partial x} + \lambda_2 \frac{\partial g(x)}{\partial x} = 0$$
$$h(x) = 0$$
$$g(x) + s^2 = 0$$
$$2s\lambda_2 = 0$$

Without specifying λ_0, the answer is not distinct.
The multipliers scale with λ_0

Choices for Regularizing λ_0

$$\lambda_0 \frac{\partial f(x)}{\partial x} + \lambda_1 \frac{\partial h(x)}{\partial x} + \lambda_2 \frac{\partial g(x)}{\partial x} = 0$$

$$h(x) = 0$$

$$g(x) + s^2 = 0$$

$$2s\lambda_2 = 0$$

Choose

$$\lambda_0 = 1 \quad \text{KKT conditions}$$

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 = 1 \quad \text{spherical constraint on multipliers}$$

Our paper uses this.

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + x^2 + s^2 = c \quad \begin{cases} \text{bounded feasible space} \\ c \text{ large enough} \end{cases}$$

Next example uses this.

$$v^T A v = c \quad \begin{cases} A \text{ positive semidefinite} \\ \text{bounded feasible space} \\ c \text{ large enough} \\ v = [\lambda_0 \ \lambda_1 \ \lambda_2 \ x \ s\ ]^T \end{cases}$$

With an appropriate choice, we can construct a base ellipse using the first-order conditions!

Small Example

Problem Statement

$$\min_{x,y} \; \lambda_0(2x - 3y)$$

$$\text{s.t. } x(x-1) = 0$$

$$y(y-1) = 0$$

First-Order Conditions

$$2\lambda_0 + 2x\lambda_1 - \lambda_1 = 0$$

$$-3\lambda_0 + 2y\lambda_2 - \lambda_2 = 0$$

$$x^2 - x = 0$$

$$y^2 - y = 0$$

Regularize $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + x^2 + y^2 = 3$

Each equation can be written in the form

$$v^T M_i v + c_i^T v = b_i$$

$$v = [\lambda_0 \; \lambda_1 \; \lambda_2 \; x \; y \; s \;]^T$$

Small Example

$$v^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} v + \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \end{bmatrix} v = 0$$

$$v^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} v + \begin{bmatrix} -3 & 0 & -1 & 0 & 0 \end{bmatrix} v = 0$$

$$v^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} v + \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \end{bmatrix} v = 0$$

$$v^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} v + \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \end{bmatrix} v = 0$$

$$v^T \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} v = 3$$

Small Example

Form base ellipse

$$v^T \underbrace{\begin{bmatrix} 1.3333 & 0 & 0 & 0 & 0 \\ 0 & 1.3333 & 0 & 1.0000 & 0 \\ 0 & 0 & 1.3333 & 0 & 1.0000 \\ 0 & 1.0000 & 0 & 2.3333 & 0 \\ 0 & 0 & 1.0000 & 0 & 2.3333 \end{bmatrix}}_{M_0} v + \underbrace{\begin{bmatrix} 1 & -1 & -1 & -1 & -1 \end{bmatrix}}_{c_0} v = 4$$

The elliptical form for the first-order conditions is

$$v^T A_i v + d_i v = b_i'$$

where

$$A_i = M_0 + \epsilon M_i$$

$$d_i = c_0 + \epsilon c_i$$

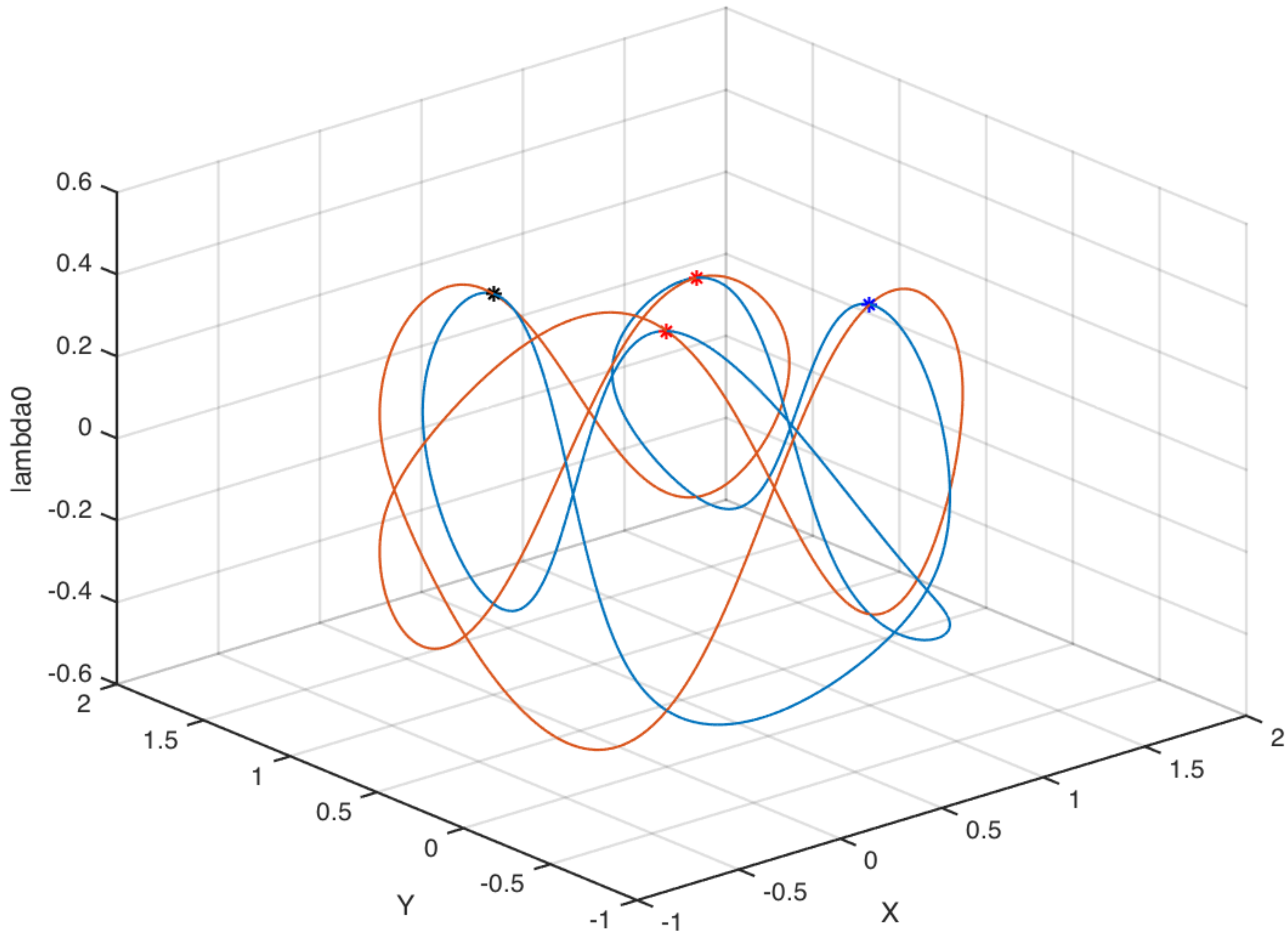
$$b_i' = 4 + \epsilon b_i$$

Small Example Results

The traces find the following four solutions (with positive λ_0):

λ_0	0.3780	0.3780	0.4629	0.2673
λ_1	-1.1339	1.1339	1.3887	-0.8018
λ_2	-0.7559	0.7559	-0.9258	0.5345
x	1	0	0	1
y	0	1	0	1
cost	2	-3	0	-1

Small Example Results



Power System Examples

We've tested this approach on several small systems, including two for which the SDP fails to find a solution.

Case9mod: 9 bus example, with 4 local minima.
Tracing results: 2003 traces find 22 critical points including all 4 known local minima.

2003 traces is a lot. We modified the approach to use a greedy search algorithm. At each new lower cost point, an additional constraint is added to find a lower cost solution. Using this monotone decreasing search, the global solution is found in **22 traces**.

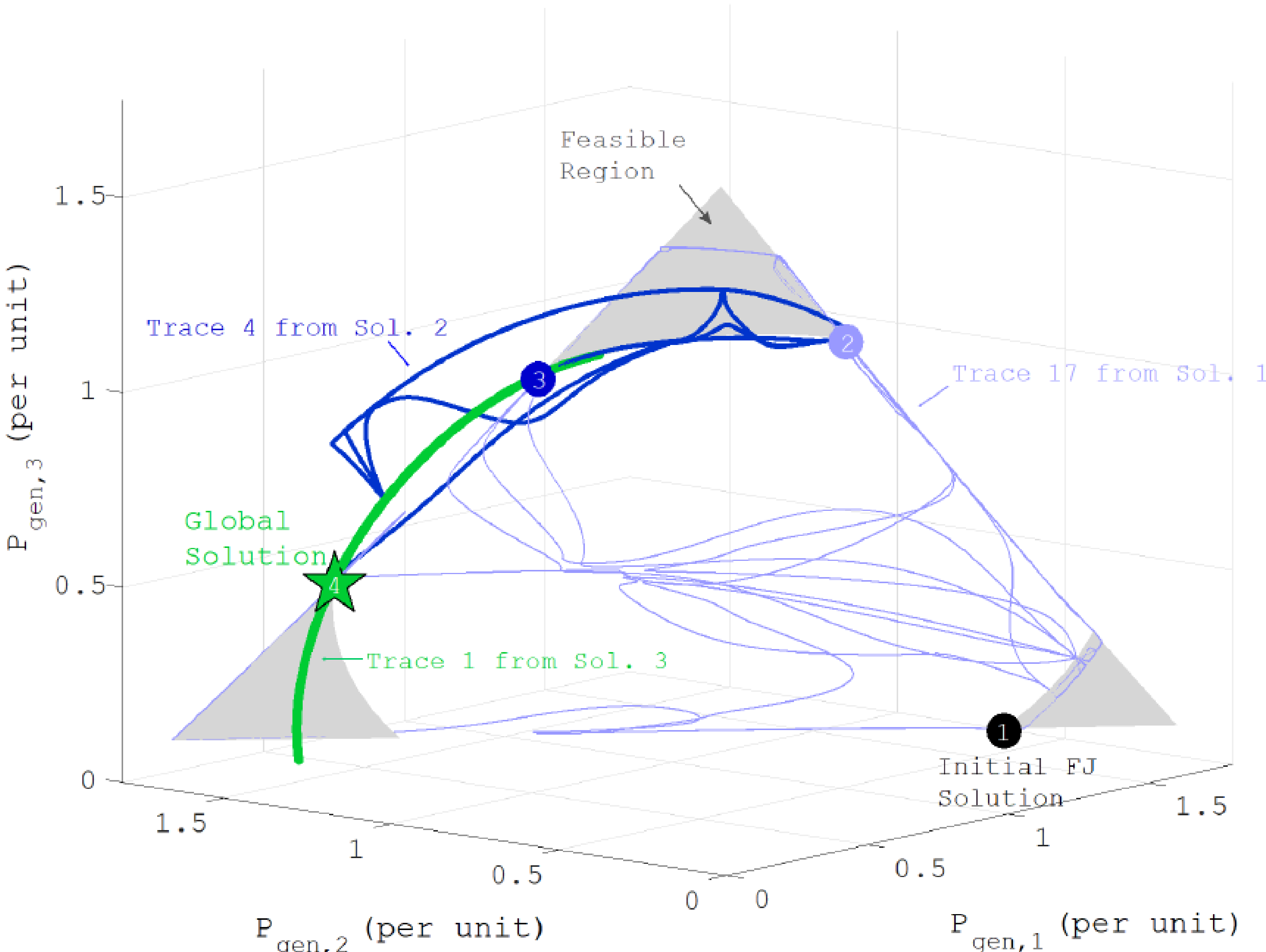


Figure courtesy of Dan Molzahn

Power System Examples

WB5: 5 bus example, with 2 local minima.
Tracing results: 628 traces find 12 critical points including the 2 local minima.

We modified the approach to use a greedy search algorithm. At each new lower cost point, an additional constraint is added to find a lower cost solution. Using this monotone decreasing search, the global solution is found in **11 traces**.

WB5bus network

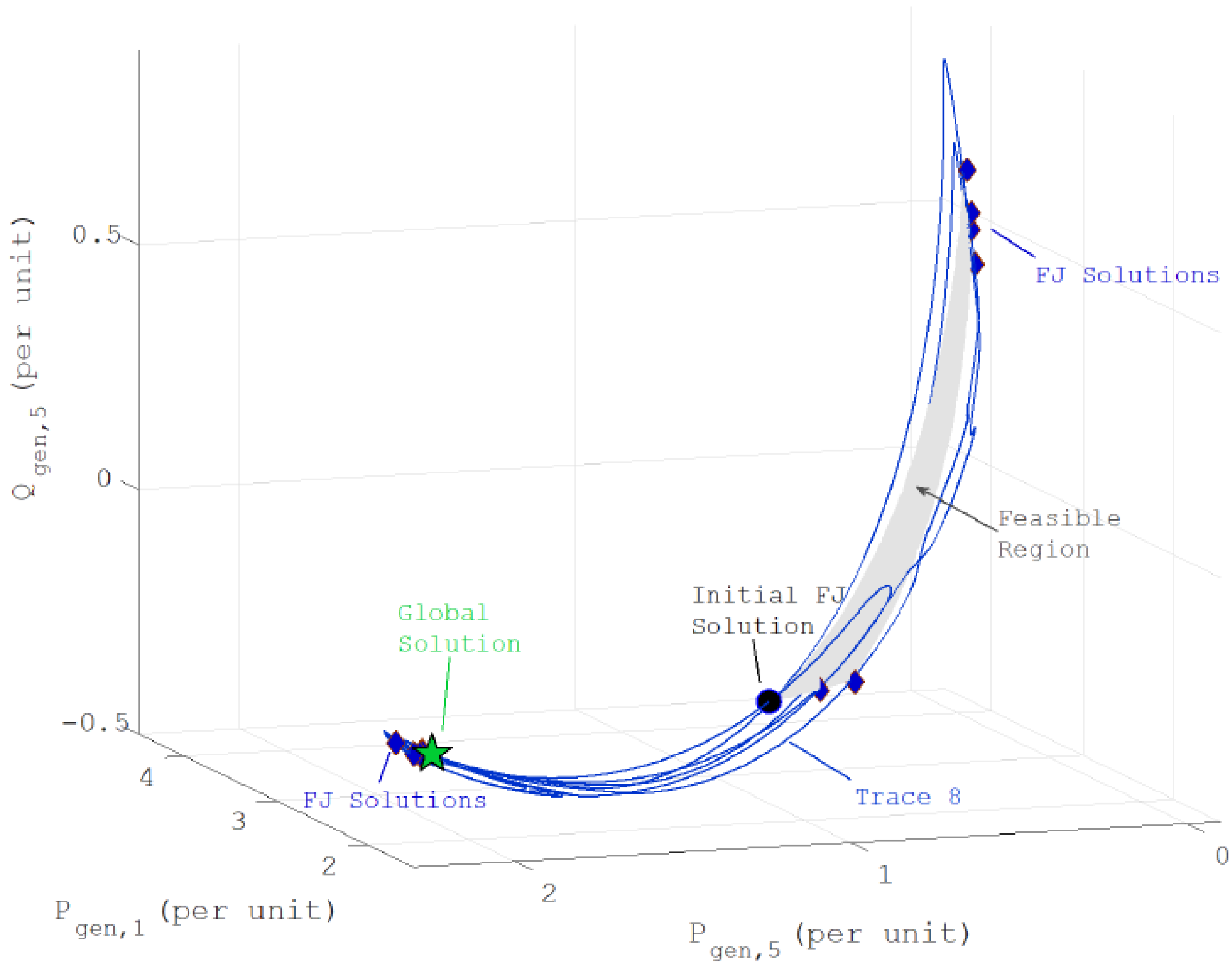
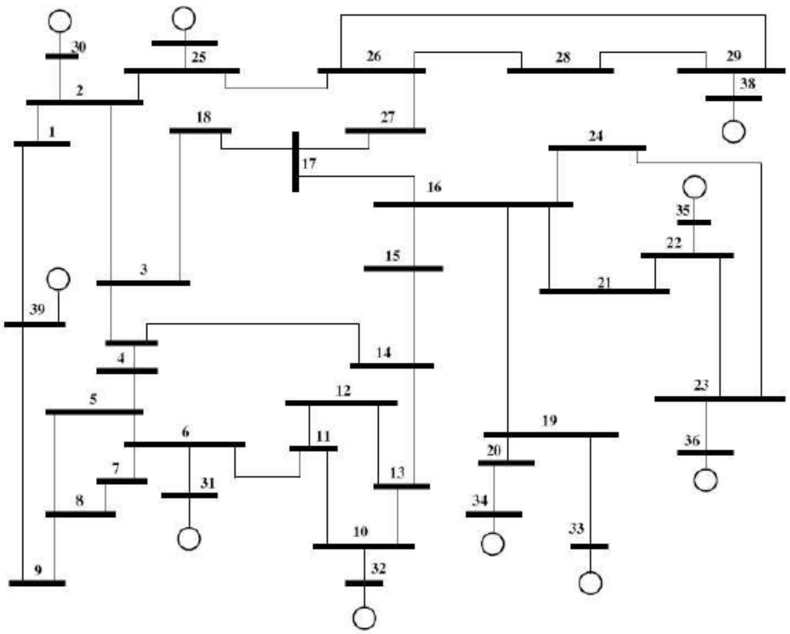


Figure courtesy of Dan Molzahn

Case39mod4



A 39-bus example for which **3 local minima** have been reported. Using our method we have found **four additional local minima**. There may be more.

Bukhsh, Waqquas A., et al. "Local solutions of the optimal power flow problem." *IEEE Transactions on Power Systems* 28.4 (2013): 4780-4788.

K. M. W A Bukhsh, A Grothey and P. A. Trodden, “Test case archive of optimal power flow (opf) problems with local optima,” [Online]. Available, http://www.maths.ed.ac.uk/optenergy/LocalOpt/.

Conclusions

- The power flow and optimal power flow problems can be cast as a problem of intersecting high-dimensional ellipses.
- The tracing method can locate solutions in different parts of a disjoint feasible set.
- Certain tracing methods scale with the number of real-valued solutions.
- A greedy search can help to focus traces towards better solutions, instead of searching for all solutions.

- Applications include problems that require finding multiple solutions to the power flow problem, and for finding multiple critical points of the optimal power flow problem.
- The method is generally applicable to Quadratically Constrained Quadratic Programs (QCQP).

Questions?

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